Problem 2B.1

Different choice of coordinates for the falling film problem. Rederive the velocity profile and the average velocity in §2.2, by replacing x by a coordinate $\bar{x}$ measured away from the wall; that is, $\bar{x} = 0$ is the wall surface, and $\bar{x} = \delta$ is the liquid–gas interface. Show that the velocity distribution is then given by

$$v_z = (\rho g \delta^2/\mu)\left[(\bar{x}/\delta) - \frac{1}{2}(\bar{x}/\delta)^2\right]\cos\beta \tag{2B.1-1}$$

and then use this to get the average velocity. Show how one can get Eq. 2B.1-1 from Eq. 2.2-18 by making a change of variable.

Solution

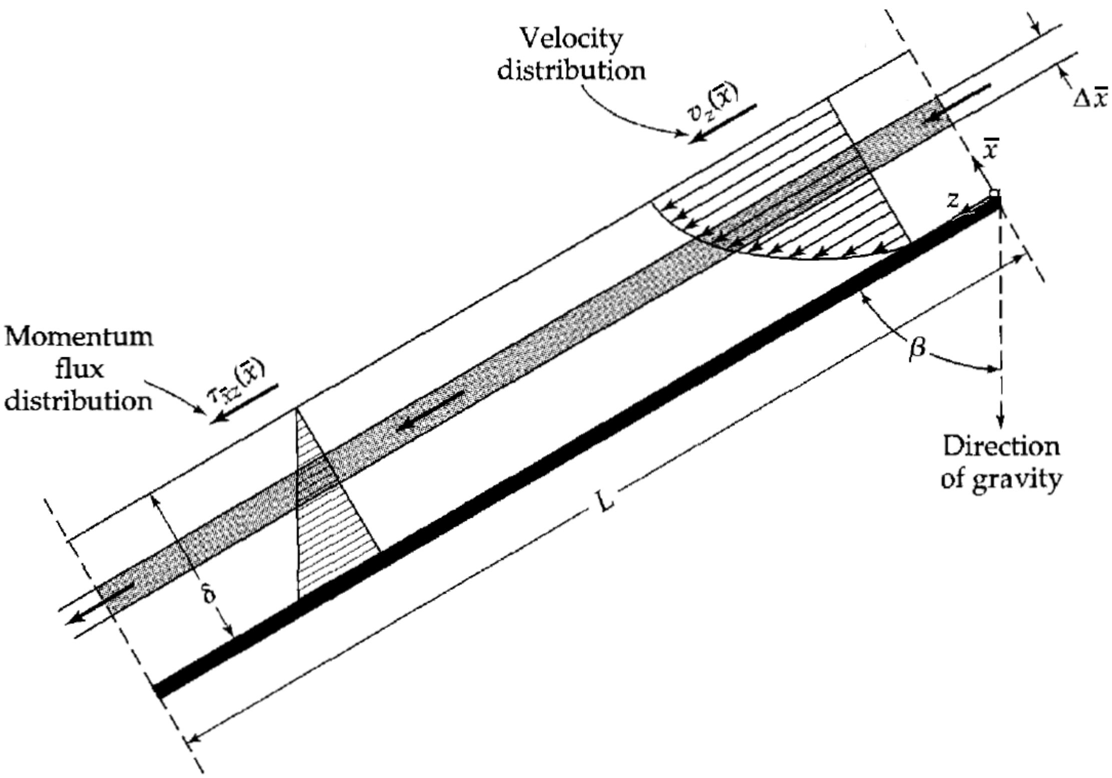

Figure 1: Schematic of the falling film with $\bar{x}$ measuring the distance from the wall.

We assume that the fluid flows in the z-direction and that its velocity varies in the x-direction.

$$v_z = v_z(\bar{x})$$

As a result, only $\phi_{\bar{x}z}$ (the z-momentum perpendicular to the $\bar{x}$-direction) and ϕ_{zz} (the z-momentum perpendicular to the z-direction) contribute to the momentum balance. We also assume that the pressure only varies with depth.

$$p = p(\bar{x})$$

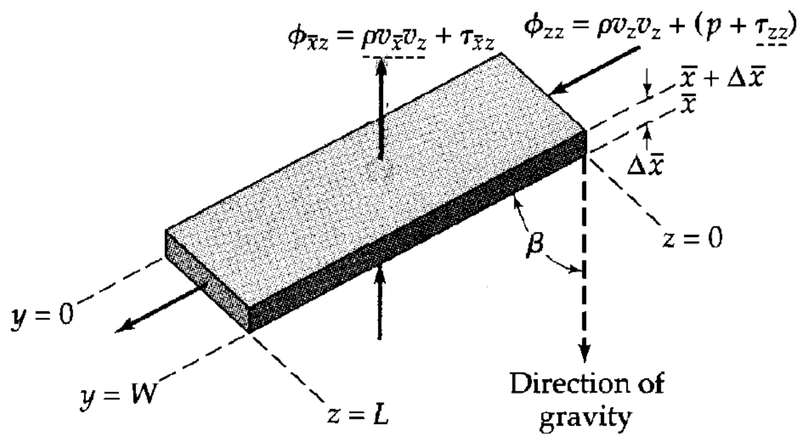

Figure 2: This is the shell over which the momentum balance is made for the falling film. As a result of the assumption $v_z = v_z(\bar{x})$, the dashed-underlined terms are equal to zero.

Rate of z-momentum into the shell at $z = 0$:	$(W\Delta\bar{x})\phi_{zz}\|_{z=0}$
Rate of z-momentum out of the shell at $z = L$:	$(W\Delta\bar{x})\phi_{zz}\|_{z=L}$
Rate of z-momentum into the shell at $\bar{x}$:	$(WL)\phi_{\bar{x}z}\|_{\bar{x}}$
Rate of z-momentum out of the shell at $\bar{x} + \Delta\bar{x}$:	$(WL)\phi_{\bar{x}z}\|_{\bar{x}+\Delta\bar{x}}$
Component of gravitational force on the shell in z-direction:	$(WL\Delta\bar{x})\rho g \cos\beta$

If we assume steady flow, then the momentum balance is

$$\text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = \mathbf{0}.$$

Considering only the z-component, we have

$$(W\Delta\bar{x})\phi_{zz}|_{z=0} - (W\Delta\bar{x})\phi_{zz}|_{z=L} + (WL)\phi_{\bar{x}z}|_{\bar{x}} - (WL)\phi_{\bar{x}z}|_{\bar{x}+\Delta\bar{x}} + (WL\Delta\bar{x})\rho g \cos\beta = 0.$$

Factor the left side.

$$W\Delta\bar{x}(\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}) + WL(\phi_{\bar{x}z}|_{\bar{x}} - \phi_{\bar{x}z}|_{\bar{x}+\Delta\bar{x}}) + WL\Delta\bar{x}\rho g \cos\beta = 0$$

Divide both sides by the volume of the shell $WL\Delta\bar{x}$.

$$\frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} - \frac{\phi_{\bar{x}z}|_{\bar{x}+\Delta\bar{x}} - \phi_{\bar{x}z}|_{\bar{x}}}{\Delta\bar{x}} + \rho g \cos\beta = 0$$

Since we assumed that the pressure and velocity only vary as a function of $\bar{x}$, ϕ_{zz} is the same at $z = 0$ as it is at $z = L$.

$$-\frac{\phi_{\bar{x}z}|_{\bar{x}+\Delta\bar{x}} - \phi_{\bar{x}z}|_{\bar{x}}}{\Delta\bar{x}} + \rho g \cos\beta = 0$$

Now take the limit as $\Delta\bar{x} \to 0$.

$$\lim_{\Delta\bar{x}\to 0} -\frac{\phi_{\bar{x}z}|_{\bar{x}+\Delta\bar{x}} - \phi_{\bar{x}z}|_{\bar{x}}}{\Delta\bar{x}} + \rho g \cos\beta = 0$$

This is how the first derivative is defined.

$$-\frac{d\phi_{\bar{x}z}}{d\bar{x}} + \rho g \cos\beta = 0$$

Since $\phi_{\bar{x}z} = \tau_{\bar{x}z} + \cancel{\rho v_x v_z}$, we get

$$\frac{d\tau_{\bar{x}z}}{d\bar{x}} = \rho g \cos\beta.$$

There are two boundary conditions associated with this differential equation—one at the surface of the film $\bar{x} = \delta$ and one at the wall $\bar{x} = 0$.

$$\text{B.C. 1 (free surface): } \tau_{\bar{x}z} = 0 \text{ when } \bar{x} = \delta$$

$$\text{B.C. 2 (no-slip): } v_z = 0 \text{ when } \bar{x} = 0$$

Integrate both sides of it with respect to $\bar{x}$.

$$\tau_{\bar{x}z}(\bar{x}) = \rho g(\cos\beta)\bar{x} + C_1$$

Apply the first boundary condition.

$$\tau_{\bar{x}z}(\delta) = \rho g(\cos\beta)\delta + C_1 = 0 \quad \rightarrow \quad C_1 = -\rho g(\cos\beta)\delta$$

So we have

$$\tau_{\bar{x}z}(\bar{x}) = \rho g(\cos\beta)\bar{x} - \rho g(\cos\beta)\delta.$$

From Newton's law of viscosity we know that $\tau_{\bar{x}z} = -\mu(dv_z/d\bar{x})$, so

$$-\mu\frac{dv_z}{d\bar{x}} = \rho g(\cos\beta)\bar{x} - \rho g(\cos\beta)\delta.$$

Divide both sides by $-\mu$.

$$\frac{dv_z}{d\bar{x}} = -\frac{\rho g}{\mu}(\cos\beta)\bar{x} + \frac{\rho g}{\mu}(\cos\beta)\delta$$

Integrate both sides with respect to $\bar{x}$ once more.

$$v_z(\bar{x}) = -\frac{\rho g}{2\mu}(\cos\beta)\bar{x}^2 + \frac{\rho g}{\mu}(\cos\beta)\delta\bar{x} + C_2$$

Apply the second boundary condition.

$$v_z(0) = C_2 = 0$$

So we have

$$v_z(\bar{x}) = -\frac{\rho g}{2\mu}(\cos\beta)\bar{x}^2 + \frac{\rho g}{\mu}(\cos\beta)\delta\bar{x}.$$

Factor $(\rho g\delta^2/\mu)\cos\beta$.

$$v_z(\bar{x}) = \frac{\rho g\delta^2}{\mu}\left(-\frac{\bar{x}^2}{2\delta^2} + \frac{\bar{x}}{\delta}\right)\cos\beta$$

Therefore, the velocity distribution is

$$v_z(\bar{x}) = \frac{\rho g\delta^2}{\mu}\left[\left(\frac{\bar{x}}{\delta}\right) - \frac{1}{2}\left(\frac{\bar{x}}{\delta}\right)^2\right]\cos\beta.$$

The average velocity is obtained by integrating the velocity distribution over the cross-sectional area and then dividing by that area.

$$\begin{aligned}
\langle v_z \rangle &= \frac{1}{A} \int v_z \, dA \\
&= \frac{1}{W\delta} \int_0^\delta v_z (W \, d\bar{x}) \\
&= \frac{1}{\delta} \int_0^\delta v_z \, d\bar{x} \\
&= \frac{\rho g \delta^2}{\mu \delta} \cos\beta \int_0^\delta \left[\left(\frac{\bar{x}}{\delta}\right) - \frac{1}{2}\left(\frac{\bar{x}}{\delta}\right)^2 \right] d\bar{x} \\
&= \frac{\rho g \delta}{\mu} \cos\beta \left(\frac{\bar{x}^2}{2\delta} - \frac{1}{2}\frac{\bar{x}^3}{3\delta^2} \right)\Big|_0^\delta \\
&= \frac{\rho g \delta}{\mu} \cos\beta \left(\frac{\delta^2}{2\delta} - \frac{1}{2}\frac{\delta^3}{3\delta^2} \right) \\
&= \frac{\rho g \delta}{\mu} \cos\beta \left(\frac{\delta}{3} \right)
\end{aligned}$$

Therefore, the average velocity is

$$\langle v_z \rangle = \frac{\rho g \delta^2}{3\mu} \cos\beta,$$

which is the same as Eq. 2.2-20. From Eq. 2.2-18,

$$v_z = \frac{\rho g \delta^2 \cos\beta}{2\mu} \left[1 - \left(\frac{x}{\delta}\right)^2 \right], \tag{2.2-18}$$

one can get Eq. 2B.1-1 by making the substitution $x = \delta - \bar{x}$.

$$\begin{aligned}
v_z &= \frac{\rho g \delta^2 \cos\beta}{2\mu} \left[1 - \left(\frac{\delta - \bar{x}}{\delta}\right)^2 \right] \\
&= \frac{\rho g \delta^2 \cos\beta}{2\mu} \left[1 - \left(1 - \frac{\bar{x}}{\delta}\right)^2 \right] \\
&= \frac{\rho g \delta^2 \cos\beta}{2\mu} \left[1 - \left(1 - 2\frac{\bar{x}}{\delta} + \frac{\bar{x}^2}{\delta^2}\right) \right] \\
&= \frac{\rho g \delta^2 \cos\beta}{2\mu} \left(2\frac{\bar{x}}{\delta} - \frac{\bar{x}^2}{\delta^2} \right) \\
&= \frac{\rho g \delta^2 \cos\beta}{\mu} \left(\frac{\bar{x}}{\delta} - \frac{1}{2}\frac{\bar{x}^2}{\delta^2} \right) \\
&= \frac{\rho g \delta^2}{\mu} \left[\left(\frac{\bar{x}}{\delta}\right) - \frac{1}{2}\left(\frac{\bar{x}}{\delta}\right)^2 \right] \cos\beta
\end{aligned}$$