Problem 2B.6

Flow of a film on the outside of a circular tube (see Fig. 2B.6). In a gas absorption experiment a viscous fluid flows upward through a small circular tube and then downward in laminar flow on the outside. Set up a momentum balance over a shell of thickness Δr in the film, as shown in Fig. 2B.6. Note that the "momentum in" and "momentum out" arrows are always taken in the positive coordinate direction, even though in this problem the momentum is flowing through the cylindrical surfaces in the negative r direction.

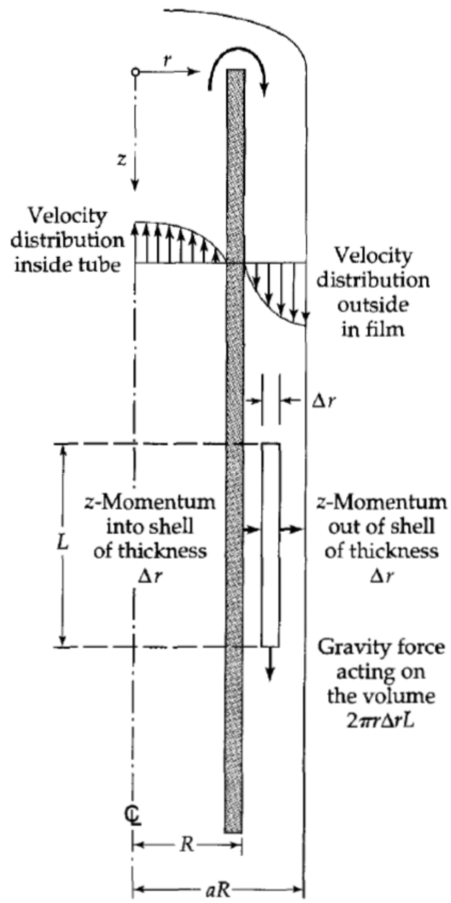

Figure 1: Fig. 2B.6 in the text. Velocity distribution and z-momentum balance for the flow of a falling film on the outside of a circular tube.

(a) Show that the velocity distribution in the falling film (neglecting end effects) is

$$v_z = \frac{\rho g R^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2 + 2a^2 \ln\left(\frac{r}{R}\right)\right] \tag{2B.6-1}$$

(b) Obtain an expression for the mass rate of flow in the film.

(c) Show that the result in (b) simplifies to Eq. 2.2-21 if the film thickness is very small.

Solution

Part (a)

We assume that the fluid flows only in the z-direction and that its velocity varies with radius r.

$$v_z = v_z(r)$$

As a result, only ϕ_{rz} (the z-momentum in the positive r-direction) and ϕ_{zz} (the z-momentum in the positive z-direction) contribute to the momentum balance. The fluid is assumed to fall because of gravity and not because of a pressure difference.

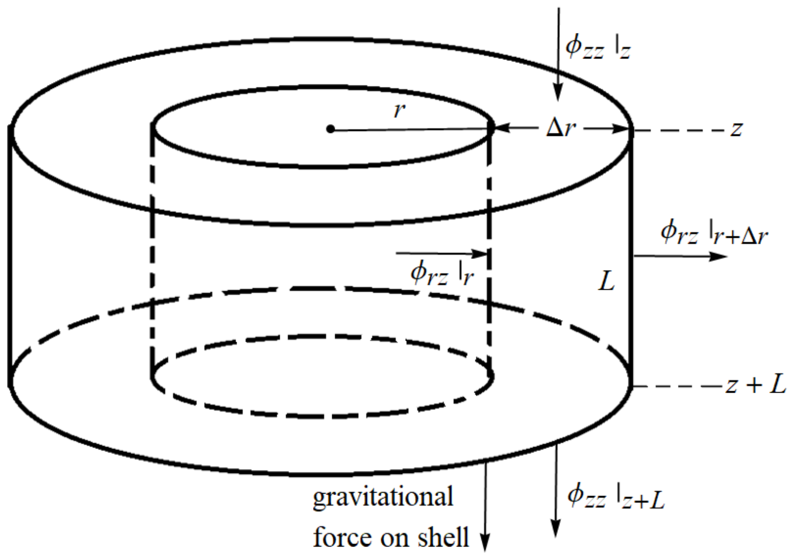

Figure 2: This is the shell over which the momentum balance is made for flow down the exterior of a cylindrical tube.

Rate of z-momentum into the shell at z:	$(2\pi r\Delta r)\phi_{zz}\|_z$
Rate of z-momentum out of the shell at $z+L$:	$(2\pi r\Delta r)\phi_{zz}\|_{z+L}$
Rate of z-momentum into the shell at r:	$(2\pi rL)\phi_{rz}\|_r$
Rate of z-momentum out of the shell at $r+\Delta r$:	$[2\pi(r+\Delta r)L]\phi_{rz}\|_{r+\Delta r}$
Component of gravitational force on the shell in z-direction:	$(2\pi r\Delta rL)\rho g$

If we assume steady flow, then the momentum balance is

$$\text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = \mathbf{0}.$$

Considering only the z-component, we have

$$(2\pi r\Delta r)\phi_{zz}|_z - (2\pi r\Delta r)\phi_{zz}|_{z+L} + (2\pi rL)\phi_{rz}|_r - [2\pi(r+\Delta r)L]\phi_{rz}|_{r+\Delta r} + (2\pi r\Delta rL)\rho g = 0.$$

Factor the left side.

$$-2\pi r\Delta r(\phi_{zz}|_{z+L} - \phi_{zz}|_z) - 2\pi L[(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r] + 2\pi r\Delta rL\rho g = 0$$

Divide both sides by $2\pi\Delta r L$.

$$-r\frac{\phi_{zz}|_{z+L}-\phi_{zz}|_z}{L}-\frac{(r+\Delta r)\phi_{rz}|_{r+\Delta r}-r\phi_{rz}|_r}{\Delta r}+\rho g r=0$$

Take the limit as $\Delta r\to 0$.

$$-r\frac{\phi_{zz}|_{z+L}-\phi_{zz}|_z}{L}-\lim_{\Delta r\to 0}\frac{(r+\Delta r)\phi_{rz}|_{r+\Delta r}-r\phi_{rz}|_r}{\Delta r}+\rho g r=0$$

The second term is the definition of the first derivative of $r\phi_{rz}$.

$$-r\frac{\phi_{zz}|_{z+L}-\phi_{zz}|_z}{L}-\frac{d}{dr}(r\phi_{rz})+\rho g r=0$$

Now substitute the expressions for ϕ_{rz} and ϕ_{zz}.

$$\begin{aligned}\phi_{rz}&=\tau_{rz}+\cancel{\rho v_r v_z}=\tau_{rz}\\ \phi_{zz}&=\cancel{p\delta_{zz}}+\cancel{\tau_{zz}}+\rho v_z v_z=\rho v_z^2\end{aligned}$$

Since v_z does not depend on z, the ρv_z^2 terms cancel.

$$-r\frac{\cancel{\rho v_z^2|_{z+L}}-\cancel{\rho v_z^2|_z}}{L}-\frac{d}{dr}(r\tau_{rz})+\rho g r=0$$

So we have

$$\frac{d}{dr}(r\tau_{rz})=\rho g r.$$

From Newton's law of viscosity we know that $\tau_{rz}=-\mu(dv_z/dr)$, so

$$\frac{d}{dr}\left(-\mu r\frac{dv_z}{dr}\right)=\rho g r.$$

At the wall $r=R$, we assume the no-slip boundary condition, and at the gas-liquid interface $r=aR$, we assume the free-surface boundary condition.

$$\begin{aligned}&\text{B.C. 1:}\quad \frac{dv_z}{dr}=0\quad\text{when}\quad r=aR\\ &\text{B.C. 2:}\quad v_z=0\quad\text{when}\quad r=R\end{aligned}$$

Integrate both sides of the differential equation with respect to r.

$$-\mu r\frac{dv_z}{dr}=\frac{\rho g}{2}r^2+C_1$$

Apply the first boundary condition now to determine C_1.

$$0=\frac{\rho g}{2}(aR)^2+C_1\quad\rightarrow\quad C_1=-\frac{\rho g a^2R^2}{2}$$

Divide both sides by $-\mu r$ to solve for dv_z/dr.

$$\frac{dv_z}{dr}=-\frac{\rho g}{2\mu}r-\frac{C_1}{\mu r}$$

Integrate both sides of the differential equation with respect to r once more.

$$v_z(r) = -\frac{\rho g}{4\mu} r^2 - \frac{C_1}{\mu} \ln r + C_2$$

Apply the second boundary condition now to determine C_2.

$$v_z(R) = -\frac{\rho g}{4\mu} R^2 - \frac{C_1}{\mu} \ln R + C_2 = 0 \quad \rightarrow \quad C_2 = \frac{\rho g}{4\mu} R^2 + \frac{C_1}{\mu} \ln R$$
$$= \frac{\rho g}{4\mu} R^2 - \frac{\rho g a^2 R^2}{2\mu} \ln R$$

With the constants of integration in hand, the velocity distribution is known.

$$\begin{aligned} v_z(r) &= -\frac{\rho g}{4\mu} r^2 + \frac{\rho g a^2 R^2}{2\mu} \ln r + \frac{\rho g}{4\mu} R^2 - \frac{\rho g a^2 R^2}{2\mu} \ln R \\ &= \frac{\rho g R^2}{4\mu} \left(-\frac{r^2}{R^2} + 2a^2 \ln r + 1 - 2a^2 \ln R \right) \\ &= \frac{\rho g R^2}{4\mu} \left[1 - \left(\frac{r}{R} \right)^2 + 2a^2 (\ln r - \ln R) \right] \end{aligned}$$

Therefore,

$$v_z = \frac{\rho g R^2}{4\mu} \left[1 - \left(\frac{r}{R} \right)^2 + 2a^2 \ln \left(\frac{r}{R} \right) \right].$$

Part (b)

The mass flow rate w is, assuming constant density ρ,

$$w = \frac{dm}{dt} = \frac{d(\rho V)}{dt} = \rho \frac{dV}{dt}.$$

The volumetric flow rate dV/dt is average velocity times cross-sectional area.

$$w = \rho \langle v_z \rangle A$$

The average velocity is obtained by integrating the velocity over the area the fluid is flowing through and then dividing by that area.

$$\begin{aligned} &= \rho \left(\frac{1}{\cancel{A}} \int v_z \, dA \right) \cancel{A} \\ &= \rho \int_R^{aR} v_z \, (2\pi r \, dr) \\ &= 2\pi\rho \int_R^{aR} r v_z \, dr \\ &= 2\pi\rho \int_R^{aR} \frac{\rho g R^2}{4\mu} r \left[1 - \left(\frac{r}{R} \right)^2 + 2a^2 \ln \left(\frac{r}{R} \right) \right] dr \end{aligned}$$

Make a substitution to solve the integral.

$$u = \frac{r}{R} \quad \rightarrow \quad r = Ru$$
$$du = \frac{dr}{R} \quad \rightarrow \quad R \, du = dr$$

The mass flow rate becomes

$$\begin{aligned}
w &= 2\pi\rho \int_1^a \frac{\rho g R^2}{4\mu}(Ru)(1-u^2+2a^2\ln u)(R\,du) \\
&= \frac{\pi\rho^2 g R^4}{2\mu}\int_1^a (u-u^3+2a^2 u\ln u)\,du \\
&= \frac{\pi\rho^2 g R^4}{2\mu}\left[\frac{u^2}{2}-\frac{u^4}{4}+2a^2\left(-\frac{u^2}{4}+\frac{u^2}{2}\ln u\right)\right]\Bigg|_1^a \\
&= \frac{\pi\rho^2 g R^4}{2\mu}\left[\frac{a^2}{2}-\frac{a^4}{4}+2a^2\left(-\frac{a^2}{4}+\frac{a^2}{2}\ln a\right)-\frac{1^2}{2}+\frac{1^4}{4}-2a^2\left(-\frac{1^2}{4}+\frac{1^2}{2}\ln 1\right)\right] \\
&= \frac{\pi\rho^2 g R^4}{2\mu}\left(\frac{a^2}{2}-\frac{a^4}{4}-\frac{a^4}{2}+a^4\ln a-\frac{1}{4}+\frac{a^2}{2}\right) \\
&= \frac{\pi\rho^2 g R^4}{2\mu}\left(a^4\ln a-\frac{3}{4}a^4+a^2-\frac{1}{4}\right)
\end{aligned}$$

Therefore,

$$w = \frac{\pi\rho^2 g R^4}{8\mu}(4a^4\ln a-3a^4+4a^2-1).$$

Part (c)

If the film thickness is very small, then we can say that a is only slightly above 1. That is, we can say $a = 1+\varepsilon$, where $0 < \varepsilon \ll 1$. Substitute this into the final answer for w.

$$w = \frac{\pi\rho^2 g R^4}{8\mu}[4(1+\varepsilon)^4\ln(1+\varepsilon)-3(1+\varepsilon)^4+4(1+\varepsilon)^2-1]$$

The Taylor series expansion for $\ln(1+\varepsilon)$ is as follows.

$$\ln(1+\varepsilon) = \varepsilon-\frac{\varepsilon^2}{2}+\frac{\varepsilon^3}{3}-\frac{\varepsilon^4}{4}+\cdots$$

Substitute this into the equation for w.

$$w = \frac{\pi\rho^2 g R^4}{8\mu}\left[4(1+\varepsilon)^4\left(\varepsilon-\frac{\varepsilon^2}{2}+\frac{\varepsilon^3}{3}-\frac{\varepsilon^4}{4}+\cdots\right)-3(1+\varepsilon)^4+4(1+\varepsilon)^2-1\right]$$

Expand the other terms.

$$= \frac{\pi\rho^2 g R^4}{8\mu}\left[(4+16\varepsilon+24\varepsilon^2+16\varepsilon^3+4\varepsilon^4)\left(\varepsilon-\frac{\varepsilon^2}{2}+\frac{\varepsilon^3}{3}-\frac{\varepsilon^4}{4}+\cdots\right)-4\varepsilon-14\varepsilon^2-12\varepsilon^3-3\varepsilon^4\right]$$

Multiply the two terms in parentheses together. Only powers up to ε^4 in the resulting series are needed.

$$\begin{aligned}
&= \frac{\pi\rho^2 g R^4}{8\mu}\left[4\varepsilon+\left(16-\frac{4}{2}\right)\varepsilon^2+\left(24-\frac{16}{2}+\frac{4}{3}\right)\varepsilon^3+\left(16-\frac{24}{2}+\frac{16}{3}-1\right)\varepsilon^4+\cdots-4\varepsilon-14\varepsilon^2-12\varepsilon^3-3\varepsilon^4\right] \\
&= \frac{\pi\rho^2 g R^4}{8\mu}\left(\cancel{4\varepsilon}+\cancel{14\varepsilon^2}+\frac{52}{3}\varepsilon^3+\frac{25}{3}\varepsilon^4+\cdots-\cancel{4\varepsilon}-\cancel{14\varepsilon^2}-12\varepsilon^3-3\varepsilon^4\right)
\end{aligned}$$

We're left with

$$\begin{aligned} w &= \frac{\pi\rho^2 g R^4}{8\mu}\left(\frac{16}{3}\varepsilon^3 + \frac{16}{3}\varepsilon^4 + \cdots\right) \\ &= \frac{\pi\rho^2 g R^4}{8\mu}\frac{16}{3}\varepsilon^3(1+\varepsilon+\cdots) \\ &= \frac{2\pi\rho^2 g\varepsilon^3 R^4}{3\mu}(1+\varepsilon+\cdots) \\ &= \frac{2\pi\rho^2 g\varepsilon^3 R^4}{3\mu} + \varepsilon\frac{2\pi\rho^2 g\varepsilon^3 R^4}{3\mu} + \cdots. \end{aligned}$$

Keep only the first term of this series for the flow rate.

$$w \approx \frac{2\pi\rho^2 g\varepsilon^3 R^4}{3\mu}$$

Comparing this with Eq. 2.2-21 on page 46,

$$w = \frac{\rho^2 g W\delta^3\cos\beta}{3\mu}, \tag{2.2-21}$$

we see that they are equivalent if $W = 2\pi R$ and $\delta = \varepsilon R$. Note that since the fluid here is flowing straight down, $\beta = 0$ and $\cos\beta = 1$.