Problem 2B.9

Low-density phenomena in compressible tube flow[2,3] (see Fig. 2B.9). As the pressure is decreased in the system studied in Example 2.3-2, deviations from Eqs. 2.3-28 and 2.3-29 arise. The gas behaves as if it slips at the tube wall. It is conventional[2] to replace the customary "no-slip" boundary condition that $v_z = 0$ at the tube wall by

$$v_z = -\zeta \frac{dv_z}{dr}, \qquad \text{at } r = R \tag{2B.9-1}$$

in which ζ is the *slip coefficient*. Repeat the derivation in Example 2.3-2 using Eq. 2B.9-1 as the boundary condition. Also make use of the experimental fact that the slip coefficient varies inversely with the pressure $\zeta = \zeta_0/p$, in which ζ_0 is a constant. Show that the mass rate of flow is

$$w = \frac{\pi(p_0 - p_L)R^4 \rho_{\text{avg}}}{8\mu L}\left(1 + \frac{4\zeta_0}{Rp_{\text{avg}}}\right) \tag{2B.9-2}$$

in which $p_{\text{avg}} = \frac{1}{2}(p_0 + p_L)$ and ρ_{avg} is the average density calculated at p_{avg}.

When the pressure is decreased further, a flow regime is reached in which the mean free path of the gas molecules is large with respect to the tube radius (*Knudsen flow*). In that regime[3]

$$w = \sqrt{\frac{2m}{\pi \kappa T}}\left(\frac{4}{3}\pi R^3\right)\left(\frac{p_0 - p_L}{L}\right) \tag{2B.9-3}$$

in which m is the molecular mass and κ is the Boltzmann constant. In the derivation of this result it is assumed that all collisions of the molecules with the solid surfaces are *diffuse* and not *specular*. The results in Eqs. 2.3-29, 2B.9-2, and 2B.9-3 are summarized in Fig. 2B.9.

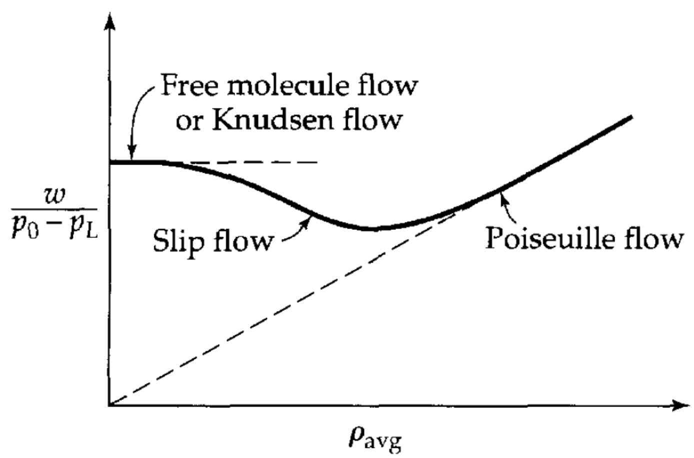

Fig. 2B.9 A comparison of the flow regimes in gas flow through a tube.

Solution

[2] E. H. Kennard, *Kinetic Theory of Gases*, McGraw-Hill, New York (1938), pp. 292-295, 300-306.

[3] M. Knudsen, *The Kinetic Theory of Gases*, Methuen, London, 3rd edition (1950). See also R. J. Silbey and R. A. Alberty, *Physical Chemistry*, Wiley, New York, 3rd edition (2001), §17.6.

This problem wants us to consider a different boundary condition than the typical no-slip one in §2.3. Up until the boundary conditions are applied, the analysis will be the same. We assume that the fluid flows only in the z-direction and that its velocity varies with radius r.

$$v_z = v_z(r)$$

As a result, only ϕ_{rz} (the z-momentum in the positive r-direction) and ϕ_{zz} (the z-momentum in the positive z-direction) contribute to the momentum balance. The pressure is assumed to vary with height z.

$$p = p(z)$$

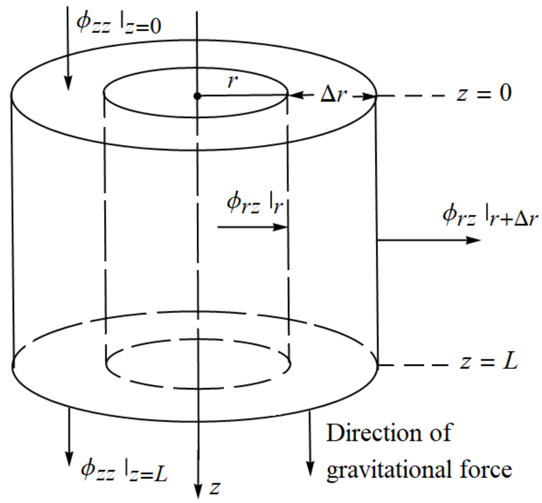

Figure 1: This is the shell over which the momentum balance is made for flow through a cylindrical tube oriented vertically.

Rate of z-momentum into the shell at $z = 0$:	$(2\pi r\Delta r)\phi_{zz}\|_{z=0}$
Rate of z-momentum out of the shell at $z = L$:	$(2\pi r\Delta r)\phi_{zz}\|_{z=L}$
Rate of z-momentum into the shell at r:	$(2\pi rL)\phi_{rz}\|_{r}$
Rate of z-momentum out of the shell at $r + \Delta r$:	$[2\pi(r+\Delta r)L]\phi_{rz}\|_{r+\Delta r}$
Component of gravitational force on the shell in z-direction:	$(2\pi r\Delta rL)\rho g$

If we assume steady flow, then the momentum balance is

$$\text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = \mathbf{0}.$$

Considering only the z-component, we have

$$(2\pi r\Delta r)\phi_{zz}|_{z=0} - (2\pi r\Delta r)\phi_{zz}|_{z=L} + (2\pi rL)\phi_{rz}|_r - [2\pi(r+\Delta r)L]\phi_{rz}|_{r+\Delta r} + (2\pi r\Delta rL)\rho g = 0.$$

Factor the left side.

$$-2\pi r\Delta r(\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}) - 2\pi L[(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r] + 2\pi r\Delta rL\rho g = 0$$

Divide both sides by $2\pi\Delta r L$.

$$-r\frac{\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}}{L} - \frac{(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r}{\Delta r} + \rho g r = 0$$

Take the limit as $\Delta r \to 0$.

$$-r\frac{\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}}{L} - \lim_{\Delta r\to 0}\frac{(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r}{\Delta r} + \rho g r = 0$$

The second term is the definition of the first derivative of $r\phi_{rz}$.

$$-r\frac{\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}}{L} - \frac{d}{dr}(r\phi_{rz}) + \rho g r = 0$$

Now substitute the expressions for ϕ_{rz} and ϕ_{zz}.

$$\begin{aligned}\phi_{rz} &= \tau_{rz} + \cancel{\rho v_r v_z} = \tau_{rz}\\ \phi_{zz} &= p\delta_{zz} + \cancel{\tau_{zz}} + \rho v_z v_z = p(z) + \rho v_z^2\end{aligned}$$

Since v_z does not depend on z, the ρv_z^2 terms cancel.

$$-r\frac{p|_{z=L} + \cancel{\rho v_z^2|_{z=L}} - p|_{z=0} - \cancel{\rho v_z^2|_{z=0}}}{L} - \frac{d}{dr}(r\tau_{rz}) + \rho g r = 0$$

Make it so that $\rho g r$ is in the fraction.

$$-r\frac{p|_{z=L} - p|_{z=0} - \rho g L}{L} - \frac{d}{dr}(r\tau_{rz}) = 0$$

Place $\rho g 0$ in the numerator.

$$-r\frac{(p|_{z=L} - \rho g L) - (p|_{z=0} - \rho g 0)}{L} - \frac{d}{dr}(r\tau_{rz}) = 0$$

The point of doing this is that now we can use the modified pressure $\mathscr{P}_z = p(z) - \rho g z$.

$$-r\frac{\mathscr{P}_L - \mathscr{P}_0}{L} - \frac{d}{dr}(r\tau_{rz}) = 0$$

So we have

$$\frac{d}{dr}(r\tau_{rz}) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L}r.$$

From Newton's law of viscosity we know that $\tau_{rz} = -\mu(dv_z/dr)$, so

$$\frac{d}{dr}\left(-\mu r\frac{dv_z}{dr}\right) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L}r.$$

One boundary condition is obtained from the assumption that the velocity is maximum furthest from the wall (at $r = 0$). The second boundary condition is provided in the problem statement at the wall $r = R$.

$$\begin{aligned}&\text{B.C. 1}: \quad \frac{dv_z}{dr} = 0, \qquad \text{at } r = 0\\ &\text{B.C. 2}: \quad v_z = -\zeta\frac{dv_z}{dr}, \qquad \text{at } r = R\end{aligned}$$

Integrate both sides of the differential equation with respect to r.

$$-\mu r \frac{dv_z}{dr} = \frac{\mathscr{P}_0 - \mathscr{P}_L}{2L} r^2 + C_1$$

Apply the first boundary condition here to determine C_1.

$$-\mu(0) \left. \frac{dv_z}{dr} \right|_{r=0} = \frac{\mathscr{P}_0 - \mathscr{P}_L}{2L} (0)^2 + C_1 \quad \rightarrow \quad 0 = C_1$$

Divide both sides by $-\mu r$ to solve for dv_z/dr.

$$\frac{dv_z}{dr} = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L} r$$

Integrate both sides of the differential equation with respect to r once more.

$$v_z(r) = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{4\mu L} r^2 + C_2$$

Apply the second boundary condition here to determine C_2.

$$v_z(R) = -\zeta \left. \frac{dv_z}{dr} \right|_{r=R}$$

$$-\frac{\mathscr{P}_0 - \mathscr{P}_L}{4\mu L} R^2 + C_2 = \zeta \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L} R \quad \rightarrow \quad C_2 = \frac{\mathscr{P}_0 - \mathscr{P}_L}{4\mu L} R(R + 2\zeta)$$

With the constants of integration in hand, the velocity distribution is known.

$$\begin{aligned} v_z(r) &= -\frac{\mathscr{P}_0 - \mathscr{P}_L}{4\mu L} r^2 + \frac{\mathscr{P}_0 - \mathscr{P}_L}{4\mu L} R(R + 2\zeta) \\ &= \frac{\mathscr{P}_0 - \mathscr{P}_L}{4\mu L} [R(R + 2\zeta) - r^2] \end{aligned}$$

The mass flow rate w is, assuming constant density ρ,

$$w = \frac{dm}{dt} = \frac{d(\rho V)}{dt} = \rho \frac{dV}{dt}.$$

The volumetric flow rate dV/dt is average velocity times cross-sectional area.

$$w = \rho \langle v_z \rangle \cdot \pi R^2$$

The average velocity is obtained by integrating the velocity over the area the fluid is flowing through and then dividing by that area.

$$\begin{aligned} &= \rho \left(\frac{1}{\cancel{\pi R^2}} \int v_z \, dA \right) \cdot \cancel{\pi R^2} \\ &= \rho \int_0^R v_z \, (2\pi r \, dr) \\ &= 2\pi\rho \int_0^R r v_z \, dr \\ &= 2\pi\rho \int_0^R r \frac{\mathscr{P}_0 - \mathscr{P}_L}{4\mu L} [R(R + 2\zeta) - r^2] \, dr \end{aligned}$$

$$\begin{aligned}
w &= \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)\rho}{2\mu L} \int_0^R [R(R+2\zeta)r - r^3]\, dr \\
&= \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)\rho}{2\mu L} \left[R(R+2\zeta)\frac{r^2}{2} - \frac{r^4}{4} \right]\Bigg|_0^R \\
&= \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)\rho}{2\mu L} \left[R(R+2\zeta)\frac{R^2}{2} - \frac{R^4}{4} \right] \\
&= \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)\rho}{2\mu L} \left(R^3\zeta + \frac{R^4}{4} \right)
\end{aligned}$$

So we have for the mass flow rate,

$$w = \frac{\pi\rho}{8\mu} \left(-\frac{\mathscr{P}_L - \mathscr{P}_0}{L} \right) (R^4 + 4R^3\zeta).$$

In the case of compressible flow (flow of a gas), the weight can be assumed to be negligible.

$$w = \frac{\pi\rho}{8\mu} \left(-\frac{p_L - p_0}{L} \right) (R^4 + 4R^3\zeta)$$

Take the limit as the length of the cylinder goes to 0.

$$w = \frac{\pi\rho}{8\mu} \left(-\lim_{L\to 0} \frac{p_L - p_0}{L} \right) (R^4 + 4R^3\zeta)$$

The term in parentheses is the first derivative of p with respect to z.

$$w = \frac{\pi\rho}{8\mu} \left(-\frac{dp}{dz} \right) (R^4 + 4R^3\zeta)$$

The density ρ can be written in terms of the pressure p using the ideal gas law,

$$pV = nR_gT,$$

where R_g represents the gas constant.

$$pV = \frac{m}{N} R_gT,$$

where m is the mass and N is the molar mass (a constant). Divide both sides by V.

$$p = \frac{\rho}{N} R_gT$$

Since the flow is assumed to be isothermal, the temperature T is also constant. This equation holds for every value of z, so

$$\begin{aligned}
&\text{At } z: && p = \frac{\rho}{N} R_gT \\
&\text{At } z = 0: && p_0 = \frac{\rho_0}{N} R_gT.
\end{aligned}$$

Divide the first equation by the second one to get a relationship between the pressure and density at $z = 0$ and the pressure and density at z. This can be solved for ρ in terms of p.

$$\frac{p}{p_0} = \frac{\rho}{\rho_0} \quad \rightarrow \quad \rho = \frac{\rho_0}{p_0} p$$

Substitute this into the equation for w.

$$w = \frac{\pi\rho_0}{8\mu p_0}\left(-p\frac{dp}{dz}\right)(R^4 + 4R^3\zeta)$$

Make use of the experimental fact that the slip coefficient varies inversely with pressure $\zeta = \zeta_0/p$, where ζ_0 is a constant.

$$w = \frac{\pi\rho_0}{8\mu p_0}\left(-p\frac{dp}{dz}\right)\left(R^4 + 4R^3\frac{\zeta_0}{p}\right)$$

Solve this differential equation by separation of variables.

$$w\,dz = \frac{\pi\rho_0}{8\mu p_0}(-p\,dp)\left(R^4 + 4R^3\frac{\zeta_0}{p}\right)$$

Integrate both sides.

$$\int_0^L w\,dz = \int_{p_0}^{p_L}\frac{\pi\rho_0}{8\mu p_0}(-p\,dp)\left(R^4 + 4R^3\frac{\zeta_0}{p}\right)$$

The mass flow rate is assumed to be constant at every height z.

$$w\int_0^L dz = -\int_{p_0}^{p_L}\frac{\pi\rho_0}{8\mu p_0}(R^4 p + 4R^3\zeta_0)\,dp$$

Proceed with the integration.

$$wL = -\frac{\pi\rho_0}{8\mu p_0}\left(R^4\frac{p^2}{2} + 4R^3\zeta_0 p\right)\Bigg|_{p_0}^{p_L}$$

Divide both sides by L to solve for w.

$$\begin{aligned}
w &= -\frac{\pi\rho_0}{8\mu L p_0}\left(R^4\frac{p_L^2}{2} + 4R^3\zeta_0 p_L - R^4\frac{p_0^2}{2} - 4R^3\zeta_0 p_0\right) \\
&= -\frac{\pi\rho_0}{8\mu L p_0}\left[\frac{R^4}{2}(p_L^2 - p_0^2) + 4R^3\zeta_0(p_L - p_0)\right] \\
&= -\frac{\pi\rho_0}{8\mu L p_0}\left[\frac{R^4}{2}(p_L - p_0)(p_L + p_0) + 4R^3\zeta_0(p_L - p_0)\right] \\
&= -\frac{\pi\rho_0}{8\mu L p_0}\left[R^4(p_L - p_0)p_{\text{avg}} + 4R^3\zeta_0(p_L - p_0)\right] \\
&= -\frac{\pi(p_L - p_0)p_{\text{avg}}\rho_0 R^4}{8\mu L p_0}\left(1 + \frac{4\zeta_0}{Rp_{\text{avg}}}\right)
\end{aligned}$$

The equation relating the pressure to density also holds for the averages.

$$\frac{p}{p_0} = \frac{\rho}{\rho_0} \quad \rightarrow \quad \rho = \frac{\rho_0}{p_0}p \quad \rightarrow \quad \rho_{\text{avg}} = \frac{\rho_0}{p_0}p_{\text{avg}}$$

Therefore,

$$w = \frac{\pi(p_0 - p_L)\rho_{\text{avg}}R^4}{8\mu L}\left(1 + \frac{4\zeta_0}{Rp_{\text{avg}}}\right),$$

where $p_{\text{avg}} = \frac{1}{2}(p_0 + p_L)$ and ρ_{avg} is the average density calculated at p_{avg}.