# Problem 2C.2

**Residence time distribution in tube flow.** Define the *residence time function* $F(t)$ to be that fraction of the fluid flowing in a conduit which flows completely through the conduit in a time interval $t$. Also define the *mean residence time* $t_m$ by the relation

$$t_m = \int_0^1 t\,dF \tag{2C.2-1}$$

**(a)** An incompressible Newtonian liquid is flowing in a circular tube of length $L$ and radius $R$, and the average flow velocity is $\langle v_z \rangle$. Show that

$$F(t) = 0 \qquad \text{for } t \le (L/2\langle v_z \rangle) \tag{2C.2-2}$$

$$F(t) = 1 - (L/2\langle v_z \rangle t)^2 \qquad \text{for } t \ge (L/2\langle v_z \rangle) \tag{2C.2-3}$$

**(b)** Show that $t_m = (L/\langle v_z \rangle)$.

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**Solution**

**Part (a)**

Consider the cross-section of fluid in a circular cylinder at the entrance $z = 0$. The fluid at the center flows fastest since it is furthest from the walls, so it will reach the cylinder's exit $z = L$ first. $F(t)$ will only start to increase from 0 at this point. The amount of time it takes for the fluid at the entrance's center to reach the exit's center is

$$t = \frac{\text{distance}}{\text{speed}} = \frac{L}{v_{z,\max}}.$$

At the moment we can say that

$$F(t) = \begin{cases} 0 & t \le \frac{L}{v_{z,\max}} \\ \text{To be determined} & t \ge \frac{L}{v_{z,\max}} \end{cases}.$$

Consider now the fluid at a radius $r$ $(0 < r < R)$ at the entrance. By the time it arrives at the exit, all the fluid with a smaller radius will have exited, whereas the fluid with larger radius will still be inside the cylinder. We can get the fraction of fluid that has exited by dividing the volumetric flow rate through a circle of radius $r$ by the total volumetric flow rate through the cylinder.

$$\begin{aligned} F(r) &= \frac{\left.\frac{dV}{dt}\right|_r}{\left.\frac{dV}{dt}\right|_R} \\ &= \frac{\int_0^r v_z(2\pi r\,dr)}{\int_0^R v_z(2\pi r\,dr)} \\ &= \frac{\int_0^r r v_z\,dr}{\int_0^R r v_z\,dr} \end{aligned}$$

For the flow in a cylindrical tube, the velocity is

$$v_z(r) = v_{z,\max}\left[1 - \left(\frac{r}{R}\right)^2\right].$$

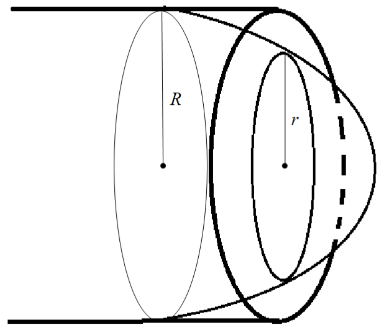


Figure 1: This is an illustration of fluid that has exited the cylinder. Here $t > L/v_{z,\max}$ and $F(t) > 0$.

Substitute this result for $v_z$.

$$F(r) = \frac{\int_0^r r v_{z,\max}\left[1 - \left(\frac{r}{R}\right)^2\right]\,dr}{\int_0^R r v_{z,\max}\left[1 - \left(\frac{r}{R}\right)^2\right]\,dr}$$

Cancel $v_{z,\max}$, multiply the numerator and denominator by $R^2$, and expand the integrands.

$$\begin{aligned}
&= \frac{\int_0^r (rR^2 - r^3)\,dr}{\int_0^R (rR^2 - r^3)\,dr} \\
&= \frac{\left(\frac{r^2R^2}{2} - \frac{r^4}{4}\right)\Big|_0^r}{\left(\frac{r^2R^2}{2} - \frac{r^4}{4}\right)\Big|_0^R} \\
&= \frac{\frac{r^2R^2}{2} - \frac{r^4}{4}}{\frac{R^4}{4}} \\
&= 2\left(\frac{r}{R}\right)^2 - \left(\frac{r}{R}\right)^4
\end{aligned}$$

Here we have $F$ as a function of $r$. Now we will get it in terms of $t$. The amount of time it takes for fluid to go from the entrance to the exit at a radius of $r$ is

$$\begin{aligned}
t &= \frac{L}{v_z(r)} \\
t &= \frac{L}{v_{z,\max}\left[1 - \left(\frac{r}{R}\right)^2\right]}
\end{aligned}$$

Solve this equation for $(r/R)^2$.

$$1 - \left(\frac{r}{R}\right)^2 = \frac{L}{v_{z,\max}t} \quad \rightarrow \quad \left(\frac{r}{R}\right)^2 = 1 - \frac{L}{v_{z,\max}t}$$

Substitute this result into $F(r)$ to get $F(t)$.

$$\begin{aligned} F(t) &= 2\left(1 - \frac{L}{v_{z,\max}t}\right) - \left(1 - \frac{L}{v_{z,\max}t}\right)^2 \\ &= 2 - 2\frac{L}{v_{z,\max}t} - 1 + 2\frac{L}{v_{z,\max}t} - \left(\frac{L}{v_{z,\max}t}\right)^2 \\ &= 1 - \left(\frac{L}{v_{z,\max}t}\right)^2 \end{aligned}$$

The final point to note is that for flow in a circular cylinder, $v_{z,\max} = 2\langle v_z \rangle$. Therefore,

$$F(t) = \begin{cases} 0 & t \leq \frac{L}{2\langle v_z \rangle} \\ 1 - \left(\frac{L}{2\langle v_z \rangle t}\right)^2 & t \geq \frac{L}{2\langle v_z \rangle} \end{cases}.$$

## Part (b)

Since $F$ is a function of time, the chain rule can be used to rewrite the definition of $t_m$.

$$t_m = \int_0^1 t\,dF = \int_0^\infty t\frac{dF}{dt}\,dt$$

When $F = 0$, no fluid has flown completely through the conduit, so $t = 0$ is the new lower limit. Though, we could use any value for $t$ up to $(L/2\langle v_z \rangle)$, since $F$ remains 0 up to this point. When $F = 1$, all fluid has flown completely through the conduit (this happens after a really long time), so $t = \infty$ is the new upper limit. The function $F$ is defined for two separate time intervals, so the integral will have to be split into two. First find $dF/dt$ from $F(t)$.

$$F(t) = \begin{cases} 0 & \text{for } t \leq \frac{L}{2\langle v_z \rangle} \\ 1 - \left(\frac{L}{2\langle v_z \rangle t}\right)^2 & \text{for } t \geq \frac{L}{2\langle v_z \rangle} \end{cases} \quad \to \quad \frac{dF}{dt} = \begin{cases} 0 & \text{for } t \leq \frac{L}{2\langle v_z \rangle} \\ \frac{L^2}{2\langle v_z \rangle^2 t^3} & \text{for } t \geq \frac{L}{2\langle v_z \rangle} \end{cases}$$

So we have

$$\begin{aligned} t_m &= \int_0^\infty t\frac{dF}{dt}\,dt \\ &= \int_0^{\frac{L}{2\langle v_z \rangle}} t\frac{dF}{dt}\,dt + \int_{\frac{L}{2\langle v_z \rangle}}^\infty t\frac{dF}{dt}\,dt \\ &= \int_0^{\frac{L}{2\langle v_z \rangle}} t(0)\,dt + \int_{\frac{L}{2\langle v_z \rangle}}^\infty t\left(\frac{L^2}{2\langle v_z \rangle^2 t^3}\right)dt \\ &= \int_{\frac{L}{2\langle v_z \rangle}}^\infty \frac{L^2}{2\langle v_z \rangle^2 t^2}\,dt \\ &= \frac{L^2}{2\langle v_z \rangle^2}\left(-\frac{1}{t}\right)\Bigg|_{\frac{L}{2\langle v_z \rangle}}^\infty \\ &= \frac{L^2}{2\langle v_z \rangle^2}\left(\frac{2\langle v_z \rangle}{L}\right). \end{aligned}$$

Therefore,

$$t_m = \frac{L}{\langle v_z \rangle}.$$