Problem 2C.5

Falling film on a conical surface (see Fig. 2C.5).[7] A fluid flows upward through a circular tube and then downward on a conical surface. Find the film thickness as a function of the distance s down the cone.

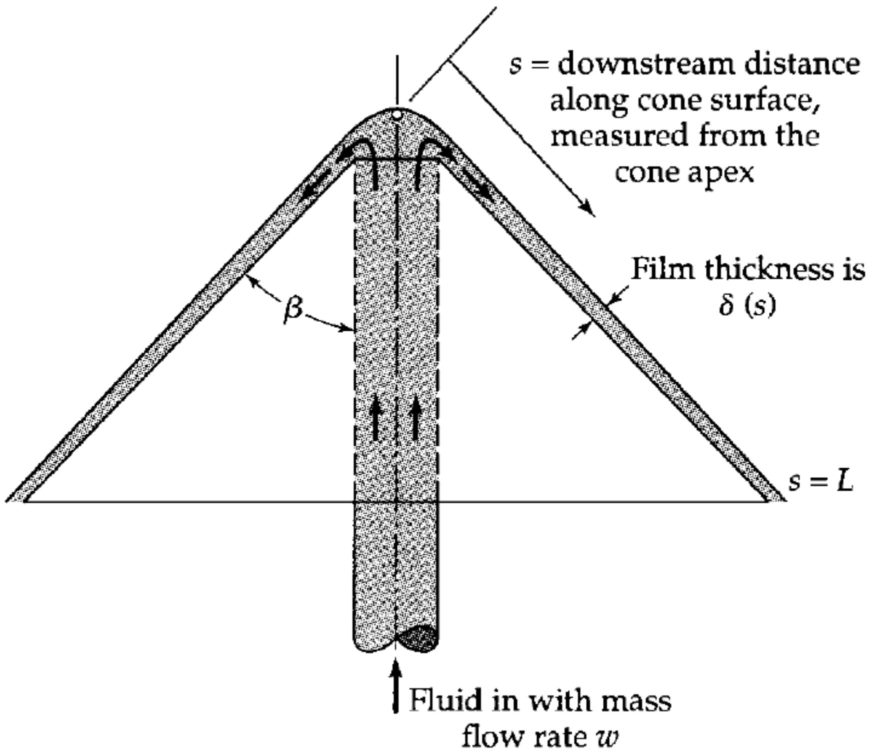

Fig. 2C.5 A falling film on a conical surface.

(a) Assume that the results of §2.2 apply *approximately* over any small region of the cone surface. Show that a mass balance on a ring of liquid contained between s and $s + \Delta s$ gives:

$$\frac{d}{ds}(s\delta\langle v\rangle) = 0 \quad \text{or} \quad \frac{d}{ds}(s\delta^3) = 0 \tag{2C.5-1}$$

(b) Integrate this equation and evaluate the constant of integration by equating the mass rate of flow w up the central tube to that flowing down the conical surface at $s = L$. Obtain the following expression for the film thickness:

$$\delta = \sqrt[3]{\frac{3\mu w}{\pi\rho^2 gL\sin 2\beta}\left(\frac{L}{s}\right)} \tag{2C.5-2}$$

Solution

[7] R. B. Bird, in *Selected Topics in Transport Phenomena*, CEP Symposium Series #58, **61**, 1-15 (1965).

Part (a)

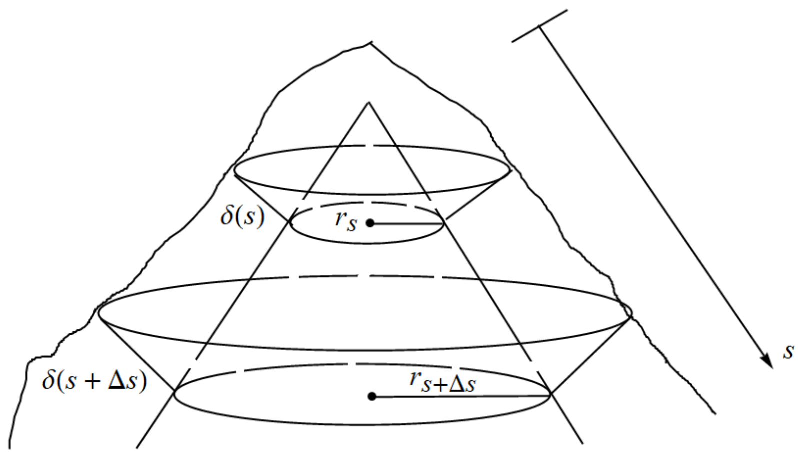

Figure 1: It's a bit difficult to visualize, but the shell under consideration here is exterior to the conical surface, above the conical frustum at $s + \Delta s$, and below the conical frustum at s.

The law of conservation of mass states that matter is neither created nor destroyed. The same amount of fluid that enters a shell per unit time must leave at that same rate; otherwise, fluid will build up or accumulate within the shell. The mathematical expression for this idea, a mass balance, is as follows.

$$\text{rate of mass in} - \text{rate of mass out} = \text{rate of mass accumulation}$$

Assuming that no fluid accumulates on the conical surface, the right-hand side is zero.

$$\text{rate of mass in} - \text{rate of mass out} = 0$$

Mass flows down the conical surface, so mass flows into the shell at s and out of the shell at $s + \Delta s$.

$$\left.\frac{dm}{dt}\right|_s - \left.\frac{dm}{dt}\right|_{s+\Delta s} = 0$$

Mass is density ρ times volume V.

$$\left.\frac{d(\rho V)}{dt}\right|_s - \left.\frac{d(\rho V)}{dt}\right|_{s+\Delta s} = 0$$

Assuming density is constant, it can be pulled out and cancelled from both sides.

$$\left.\frac{dV}{dt}\right|_s - \left.\frac{dV}{dt}\right|_{s+\Delta s} = 0$$

The volumetric flow rate is average velocity times the cross-sectional area that the fluid flows through.

$$(\langle v\rangle A)|_s - (\langle v\rangle A)|_{s+\Delta s} = 0$$

To obtain the cross-sectional area we multiply the circumference of the circle at s or $s + \Delta s$ by the film thickness $\delta(s)$ or $\delta(s + \Delta s)$, respectively. For the average velocity we use the result for a falling film on an inclined surface in §2.2.

$$\langle v \rangle = \frac{\rho g \delta^2 \cos\beta}{3\mu}$$

The mass balance becomes

$$\frac{\rho g [\delta(s)]^2 \cos\beta}{3\mu} \cdot 2\pi r_s \delta(s) - \frac{\rho g [\delta(s+\Delta s)]^2 \cos\beta}{3\mu} \cdot 2\pi r_{s+\Delta s} \delta(s+\Delta s) = 0.$$

Divide both sides by $\rho g \cos\beta \cdot 2\pi/3\mu$.

$$r_s[\delta(s)]^3 - r_{s+\Delta s}[\delta(s+\Delta s)]^3 = 0$$

The relationship between r and s is $r = s\sin\beta$, so

$$\begin{aligned} r_s &= s\sin\beta \\ r_{s+\Delta s} &= (s+\Delta s)\sin\beta. \end{aligned}$$

Substitute these results into the mass balance.

$$s\sin\beta[\delta(s)]^3 - (s+\Delta s)\sin\beta[\delta(s+\Delta s)]^3 = 0$$

Divide both sides by $-(\sin\beta)\Delta s$.

$$\frac{(s+\Delta s)[\delta(s+\Delta s)]^3 - s[\delta(s)]^3}{\Delta s} = 0$$

Take the limit as $\Delta s \to 0$.

$$\lim_{\Delta s \to 0} \frac{(s+\Delta s)[\delta(s+\Delta s)]^3 - s[\delta(s)]^3}{\Delta s} = 0$$

This is how the first derivative of $s[\delta(s)]^3$ is defined. Therefore,

$$\frac{d}{ds}(s\delta^3) = 0.$$

Part (b)

Integrate both sides of this differential equation with respect to s.

$$s\delta^3 = C_1$$

Solve this for δ^3.

$$\delta^3 = \frac{C_1}{s}$$

At $s = L$, we have

$$\delta^3 = \frac{C_1}{L}. \tag{1}$$

In §2.2 the mass flow rate w of a film falling down an incline is given in Eq. 2.2-21.

$$w = \frac{\rho^2 g W \delta^3 \cos\beta}{3\mu} \tag{2.2-21}$$

To adapt this result to a film falling down a conical surface, set the width of the film W equal to $2\pi r$, or $2\pi s \sin\beta$.

$$w = \frac{\rho^2 g (2\pi s \sin\beta)\delta^3 \cos\beta}{3\mu}$$

Use the trigonometric identity $2\sin\beta\cos\beta = \sin 2\beta$.

$$w = \frac{\pi\rho^2 g s(\sin 2\beta)\delta^3}{3\mu}$$

To determine C_1 we use the condition that the mass flow rate up the circular tube w is equal to the mass flow rate down the conical surface at $s = L$. That is,

$$\begin{aligned} w &= \left.\frac{dm}{dt}\right|_{s=L} \\ &= \left[\frac{\pi\rho^2 g s(\sin 2\beta)\delta^3}{3\mu}\right]\Bigg|_{s=L} \\ &= \frac{\pi\rho^2 g L(\sin 2\beta)\delta^3}{3\mu}. \end{aligned}$$

Solve this equation for δ^3.

$$\delta^3 = \frac{3\mu w}{\pi\rho^2 g L \sin 2\beta} \tag{2}$$

Set equations (1) and (2) equal to each other to solve for C_1.

$$\frac{C_1}{L} = \frac{3\mu w}{\pi\rho^2 g L \sin 2\beta} \quad \rightarrow \quad C_1 = \frac{3\mu w}{\pi\rho^2 g \sin 2\beta}$$

Now that C_1 is determined, the film thickness is known.

$$\begin{aligned} \delta^3 &= \frac{3\mu w}{\pi\rho^2 g \sin 2\beta}\frac{1}{s} \\ &= \frac{3\mu w}{\pi\rho^2 g L \sin 2\beta}\left(\frac{L}{s}\right) \end{aligned}$$

Therefore,

$$\delta = \sqrt[3]{\frac{3\mu w}{\pi\rho^2 g L \sin 2\beta}\left(\frac{L}{s}\right)}.$$