Problem 2C.6

Rotating cone pump (see Fig. 2C.6). Find the mass rate of flow through this pump as a function of the gravitational acceleration, the impressed pressure difference, the angular velocity of the cone, the fluid viscosity and density, the cone angle, and other geometrical quantities labeled in the figure.

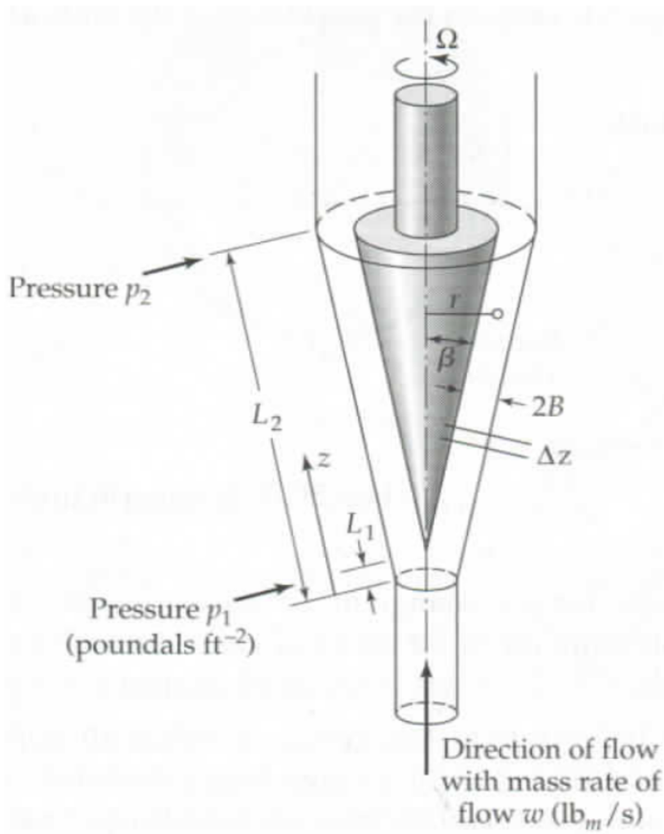

Figure 1: This is Fig. 2C.6 in the text. A rotating cone pump. The variable r is the distance from the axis of rotation out to the center of the slit.

(a) Begin by analyzing the system without the rotation of the cone. Assume that it is possible to apply the results of Problem 2B.3 locally. That is, adapt the solution for the mass flow rate from that problem by making the following replacements:

replace $(\mathscr{P}_0 - \mathscr{P}_L)/L$ by $-d\mathscr{P}/dz$

replace W by $2\pi r = 2\pi z \sin\beta$

thereby obtaining

$$w = \frac{2}{3}\left(-\frac{d\mathscr{P}}{dz}\right)\frac{B^3\rho\cdot 2\pi z\sin\beta}{\mu} \tag{2C.6-1}$$

The mass flow rate w is a constant over the range of z. Hence this equation can be integrated to give

$$(\mathscr{P}_1 - \mathscr{P}_2) = \frac{3}{4\pi}\frac{\mu w}{B^3\rho\sin\beta}\ln\frac{L_2}{L_1} \tag{2C.6-2}$$

(b) Next, modify the above result to account for the fact that the cone is rotating with angular velocity Ω. The mean centrifugal force per unit volume acting on the fluid in the slit will have a z-component *approximately* given by

$$(F_{\text{centrif}})_z = K\rho\Omega^2 z \sin^2\beta \tag{2C.6-3}$$

What is the value of K? Incorporate this as an additional force tending to drive the fluid through the channel. Show that this leads to the following expression for the mass rate of flow:

$$w = \frac{4\pi B^3\rho\sin\beta}{3\mu}\left[\frac{(\mathscr{P}_1 - \mathscr{P}_2) + \left(\frac{1}{2}K\rho\Omega^2\sin^2\beta\right)(L_2^2 - L_1^2)}{\ln(L_2/L_1)}\right] \tag{2C.6-4}$$

Here $\mathscr{P}_i = p_i + \rho g L_i \cos\beta$.

Solution

Part (a)

Without rotation the flow through the pump is essentially flow through a slit at an angle β. The mass flow rate through a slit from Problem 2B.3 is

$$w = \frac{2}{3}\frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L},$$

where $\mathscr{P} = p - \rho g z$ and z measures vertical distance. Make the indicated replacements.

$$w = \frac{2}{3}\left(-\frac{d\mathscr{P}}{dz}\right)\frac{B^3\rho\cdot 2\pi z\sin\beta}{\mu}$$

Because the positive z-direction goes from the bottom to the top of the pump, $\mathscr{P}$ here is $p + \rho g L\cos\beta$. Assuming w is constant, the differential equation can be solved by separation of variables.

$$\frac{3\mu w}{4B^3\rho\pi\sin\beta}\frac{dz}{z} = -d\mathscr{P}$$

Integrate both sides.

$$\int_{L_1}^{L_2}\frac{3\mu w}{4B^3\rho\pi\sin\beta}\frac{dz}{z} = \int_{\mathscr{P}_1}^{\mathscr{P}_2}(-d\mathscr{P})$$

Proceed with the integration.

$$\frac{3\mu w}{4B^3\rho\pi\sin\beta}(\ln L_2 - \ln L_1) = (\mathscr{P}_1 - \mathscr{P}_2)$$

Therefore,

$$(\mathscr{P}_1 - \mathscr{P}_2) = \frac{3}{4\pi}\frac{\mu w}{B^3\rho\sin\beta}\ln\frac{L_2}{L_1}.$$

Part (b)

The mean centrifugal force can be found by considering the centrifugal force of fluid at the center of the slit.

$$F_{\text{centrifugal}} = ma_{\text{centrifugal}} = m\frac{v^2}{r}$$

Since Ω is perpendicular to the radial direction, it is related to v by $v = \Omega r$. For fluid at the center of the slit a factor of 1/2 is included because it is halfway between a wall that is stationary (the container) and a wall that is moving (the cone). It is assumed that the fluid does not slip on these surfaces.

$$\begin{aligned} F_{\text{centrifugal}} &= \frac{m}{r}\left(\frac{1}{2}\Omega r\right)^2 \\ &= \frac{m}{r}\frac{\Omega^2 r^2}{4} \\ &= \frac{1}{4}m\Omega^2 r \end{aligned}$$

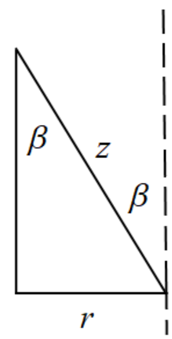

Figure 2: A relationship between r and z can be obtained from this triangle: $r = z\sin\beta$.

$$F_{\text{centrifugal}} = \frac{1}{4}m\Omega^2 z\sin\beta$$

This centrifugal force is in the radial direction. To get the component of this force in the z-direction, see Figure 3.

$$(F_{\text{centrifugal}})_z = \frac{1}{4}m\Omega^2 z\sin^2\beta$$

Finally, to get the centrifugal force in the z-direction per unit volume, divide both sides by V. Replace m/V with the density ρ.

$$(F_{\text{centrif}})_z = \frac{(F_{\text{centrifugal}})_z}{V} = \frac{1}{4}\rho\Omega^2 z\sin^2\beta$$

Therefore, $K = 1/4$. Include this force in the equation for the mass flow rate to account for the rotation of the cone.

$$\underbrace{w = \frac{2}{3}\left(-\frac{d\mathscr{P}}{dz}\right)\frac{B^3\rho\cdot 2\pi z\sin\beta}{\mu}}_{\text{no rotation}} \quad\rightarrow\quad w = \frac{2}{3}\left(-\frac{d\mathscr{P}}{dz} + \frac{1}{4}\rho\Omega^2 z\sin^2\beta\right)\frac{B^3\rho\cdot 2\pi z\sin\beta}{\mu}$$

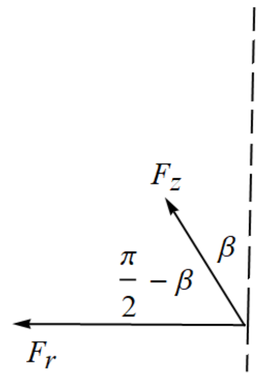

Figure 3: A relationship between a radial force F_r and a force in the z-direction F_z can be obtained from this triangle: $F_r \cos\left(\frac{\pi}{2} - \beta\right) = F_z$, or $F_r \sin\beta = F_z$.

Once again, the differential equation can be solved by separation of variables, assuming the mass flow rate w is constant.

$$\frac{3\mu w}{4\pi B^3 \rho \sin\beta} \frac{1}{z} = -\frac{d\mathscr{P}}{dz} + \frac{1}{4}\rho\Omega^2 z \sin^2\beta$$

$$\frac{d\mathscr{P}}{dz} = -\frac{3\mu w}{4\pi B^3 \rho \sin\beta} \frac{1}{z} + \frac{1}{4}\rho\Omega^2 z \sin^2\beta$$

$$d\mathscr{P} = \left(-\frac{3\mu w}{4\pi B^3 \rho \sin\beta} \frac{1}{z} + \frac{1}{4}\rho\Omega^2 z \sin^2\beta\right) dz$$

Integrate both sides.

$$\int_{\mathscr{P}_1}^{\mathscr{P}_2} d\mathscr{P} = \int_{L_1}^{L_2} \left(-\frac{3\mu w}{4\pi B^3 \rho \sin\beta} \frac{1}{z} + \frac{1}{4}\rho\Omega^2 z \sin^2\beta\right) dz$$

$$(\mathscr{P}_2 - \mathscr{P}_1) = \left(-\frac{3\mu w}{4\pi B^3 \rho \sin\beta} \ln z + \frac{1}{4}\rho\Omega^2 \frac{z^2}{2} \sin^2\beta\right)\Bigg|_{L_1}^{L_2}$$

$$(\mathscr{P}_2 - \mathscr{P}_1) = -\frac{3\mu w}{4\pi B^3 \rho \sin\beta}(\ln L_2 - \ln L_1) + \frac{1}{4}\rho\Omega^2 \frac{L_2^2 - L_1^2}{2} \sin^2\beta$$

Combine the logarithms and solve for w.

$$\frac{3\mu w}{4\pi B^3 \rho \sin\beta} \ln\frac{L_2}{L_1} = (\mathscr{P}_1 - \mathscr{P}_2) + \frac{1}{8}\rho\Omega^2 (L_2^2 - L_1^2) \sin^2\beta$$

Therefore,

$$w = \frac{4\pi B^3 \rho \sin\beta}{3\mu}\left[\frac{(\mathscr{P}_1 - \mathscr{P}_2) + \left(\frac{1}{2}K\rho\Omega^2 \sin^2\beta\right)(L_2^2 - L_1^2)}{\ln(L_2/L_1)}\right],$$

where $K = 1/4$ and $\mathscr{P}_i = p_i + \rho g L_i \cos\beta$.