Problem 2C.7

A simple rate-of-climb indicator (see Fig. 2C.7). Under the proper circumstances the simple apparatus shown in the figure can be used to measure the rate of climb of an airplane. The gauge pressure inside the Bourdon element is taken as proportional to the rate of climb. For the purposes of this problem the apparatus may be assumed to have the following properties: (i) the capillary tube (of radius R and length L, with $R \ll L$) is of negligible volume but appreciable flow resistance; (ii) the Bourdon element has a constant volume V and offers negligible resistance to the flow; and (iii) flow in the capillary is laminar and incompressible, and the volumetric flow rate depends only on the conditions at the ends of the capillary.

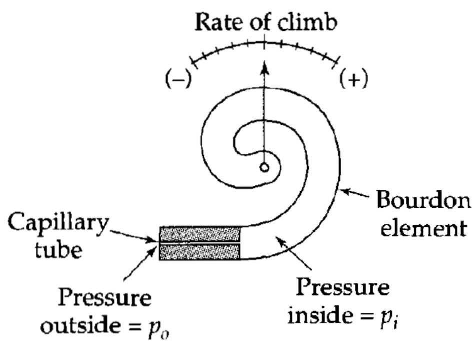

Fig. 2C.7 A rate-of-climb indicator.

(a) Develop an expression for the change of pressure with altitude, neglecting temperature changes, and considering air to be an ideal gas of constant composition. (*Hint:* Write a shell balance in which the weight of gas is balanced against the static pressure.)

(b) By making a mass balance over the gauge, develop an approximate relation between gauge pressure $p_i - p_o$ and rate of climb v_z for a long continued constant-rate climb. Neglect change of air viscosity, and assume changes in air density to be small.

(c) Develop an approximate expression for the "relaxation time" t_{rel} of the indicator—that is, the time required for the gauge pressure to drop to $1/e$ of its initial value when the external pressure is suddenly changed from zero (relative to the interior of the gauge) to some different constant value, and maintained indefinitely at this new value.

(d) Discuss the usefulness of this type of indicator for small aircraft.

(e) Justify the plus and minus signs in the figure.

Answers: **(a)** $dp/dz = -\rho g = -(pM/RT)g$

(b) $p_i - p_o \approx v_z(8\mu L/\pi R^4)(MgV/R_gT)$, where R_g is the gas constant and M is the molecular weight.

(c) $t_{\text{rel}} = (128/\pi)(\mu VL/D^4\bar{p})$, where $\bar{p} = \dfrac{1}{2}(p_i + p_o)$

Solution

Part (a)

Following the hint, a rectangular shell will be considered where forces due to gravity and pressure act on it.

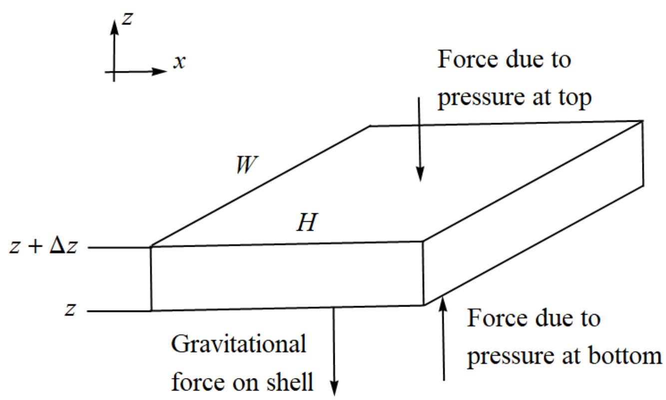

Figure 1: This is the shell over which the force balance is made to determine the change in pressure with altitude.

The positive z-direction is chosen to point upward, and the forces due to pressure act normal to the shell's surface on the top (at $z + \Delta z$) and bottom (at z).

Force due to pressure at top:	$-\, p\big\|_{z+\Delta z} \cdot HW$
Force due to pressure at bottom:	$p\big\|_{z} \cdot HW$
Gravitational force on shell:	$-\, \rho g \cdot HW\Delta z$

Assuming the shell is in equilibrium, the sum of the forces acting on it is equal to 0.

$$\sum F_z = 0$$

Place the forces above on the left side.

$$-\, p\big|_{z+\Delta z} \cdot HW + p\big|_{z} \cdot HW - \rho g \cdot HW\Delta z = 0$$

Divide both sides by $HW\Delta z$ and factor a minus sign from the first two terms.

$$-\frac{p\big|_{z+\Delta z} - p\big|_{z}}{\Delta z} - \rho g = 0$$

$$\frac{p\big|_{z+\Delta z} - p\big|_{z}}{\Delta z} = -\rho g$$

Now take the limit as $\Delta z \to 0$.

$$\lim_{\Delta z \to 0} \frac{p\big|_{z+\Delta z} - p\big|_{z}}{\Delta z} = -\rho g$$

The term on the left is how the first derivative of p with respect to z is defined.

$$\frac{dp}{dz} = -\rho g$$

If the air outside the aircraft is assumed to be an ideal gas, then the equation of state,

$$pv = nR_gT,$$

can be used to write the density ρ in terms of the pressure p. Start by writing the number of moles n as the mass m divided by the molar mass M.

$$pv = \frac{m}{M}R_gT \quad \rightarrow \quad \frac{pM}{R_gT} = \frac{m}{v} = \rho$$

Therefore,

$$\frac{dp}{dz} = -\frac{pM}{R_gT}g.$$

Part (b)

The differential equation above is for the pressure outside the aircraft.

$$\frac{dp_o}{dz} = -\frac{p_oM}{R_gT}g$$

The boundary condition that goes with it is $p_o(0) = p_s$, namely at $z = 0$ (sea level) the pressure is known to be p_s. Solve the differential equation with separation of variables.

$$\frac{dp_o}{p_o} = -\frac{Mg}{R_gT}\,dz$$

Integrate both sides.

$$\int \frac{dp_o}{p_o} = \int -\frac{Mg}{R_gT}\,dz$$

$$\ln p_o = -\frac{Mg}{R_gT}z + C$$

Exponentiate both sides.

$$p_o(z) = \exp\left(-\frac{Mg}{R_gT}z + C\right)$$

Introduce a new constant of integration.

$$p_o(z) = A\exp\left(-\frac{Mg}{R_gT}z\right)$$

Apply the boundary condition here: $p_o(0) = A = p_s$.

$$p_o(z) = p_s\exp\left(-\frac{Mg}{R_gT}z\right) \tag{1}$$

Mass flows through the capillary tube as a consequence of the pressure difference between the inside of the Bourdon element and the outside, $p_i - p_o$.

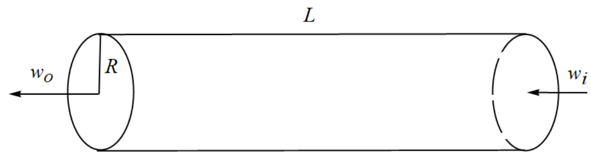

Figure 2: Schematic of the capillary tube. It is assumed that $R \ll L$.

Assuming no mass accumulates inside the tube, the following mass balance can be made.

$$\text{Rate of mass in} - \text{Rate of mass out} = 0$$

$$w_i - w_o = 0$$

Bring w_o to the right side.

$$\begin{aligned} w_i &= w_o = w \\ \left.\frac{dm}{dt}\right|_i &= \left.\frac{dm}{dt}\right|_o \\ \left.\frac{d(\rho v)}{dt}\right|_i &= \left.\frac{d(\rho v)}{dt}\right|_o \end{aligned}$$

Density is assumed to be constant.

$$\left.\rho\frac{dv}{dt}\right|_i = \left.\rho\frac{dv}{dt}\right|_o$$

Divide both sides by ρ.

$$\left.\frac{dv}{dt}\right|_i = \left.\frac{dv}{dt}\right|_o = \frac{dv}{dt}.$$

From these equations we conclude that the rate of mass flow at one end of the tube is equal to that at the other end (use w to represent it). The same can be said for the volumetric flow (use dv/dt to represent it). The fact that a capillary tube connects the Bourdon element with the outside means that the Hagen-Poiseuille equation can be used for the mass flow rate through it.

$$w = \frac{\pi(p_i - p_o)R^4\rho}{8\mu L}$$

Solve this equation for the pressure difference. For ρ an average density for air at both ends of the tube should be used. Then it cancels out.

$$\begin{aligned} p_i - p_o &= \frac{8\mu L}{\pi R^4 \rho} w \\ &= \frac{8\mu L}{\pi R^4 \rho}\rho\frac{dv}{dt} \\ &= \frac{8\mu L}{\pi R^4}\frac{dv}{dt} \end{aligned} \tag{2}$$

Our aim now is to obtain an expression for dv/dt. Use the equation of state to solve for v.

$$pv = nR_gT \quad \rightarrow \quad v = \frac{nR_gT}{p}$$

So we have

$$\begin{aligned}\frac{dv}{dt} &= \frac{d}{dt}\left(\frac{nR_gT}{p_o}\right)\\ &= nR_gT\frac{d}{dt}\left(\frac{1}{p_o}\right)\\ &= nR_gT\left(-\frac{1}{p_o^2}\right)\frac{dp_o}{dt}\\ &= nR_gT\left(-\frac{1}{p_o^2}\right)\frac{dp_o}{dz}\frac{dz}{dt},\end{aligned}$$

where dz/dt is the rate of climb of the aircraft. This is assumed to be equal to v_z, a constant. Substitute the expression for p_o in equation (1) here.

$$\begin{aligned}\frac{dv}{dt} &= nR_gT\left[\left(-\frac{1}{p_s^2}\right)\exp\left(2\frac{Mg}{R_gT}z\right)\right]\left[p_s\left(-\frac{Mg}{R_gT}\right)\exp\left(-\frac{Mg}{R_gT}z\right)\right]v_z\\ &= \frac{nR_gT}{p_s}\frac{Mg}{R_gT}\exp\left(\frac{Mg}{R_gT}z\right)v_z\\ &= V\frac{Mg}{R_gT}\exp\left(\frac{Mg}{R_gT}z\right)v_z\end{aligned}$$

Plug this expression for dv/dt back into equation (2) for the pressure difference.

$$p_i - p_o = v_z\frac{8\mu L}{\pi R^4}\frac{MgV}{R_gT}\exp\left(\frac{Mg}{R_gT}z\right)$$

Use the Taylor series expansion for the exponential function.

$$p_i - p_o = v_z\frac{8\mu L}{\pi R^4}\frac{MgV}{R_gT}\left[1+\left(\frac{Mg}{R_gT}z\right)+\left(\frac{Mg}{R_gT}z\right)^2+\cdots\right]$$

Therefore,

$$p_i - p_o \approx v_z\frac{8\mu L}{\pi R^4}\frac{MgV}{R_gT}.$$

Part (c)

Suppose initially that the outside pressure and the inside pressure are the same at p_i and that all of a sudden the outside pressure lowers to p_o and stays there. As a result of the pressure difference, mass will flow through the capillary tube to the outside. The pressure inside the Bourdon element will thus lower as a function of time and eventually come to equilibrium with the outside pressure. According to the Hagen-Poiseuille equation, the mass flow rate through the tube is

$$w = \frac{\pi[P(t)-p_o]R^4\bar{\rho}}{8\mu L},$$

where $P(t)$ represents the pressure on the inside of the Bourdon element as a function of time and $\bar{\rho}$ represents the average density of air in the tube $\bar{\rho} = (\rho_i + \rho_o)/2$. The average density is used as a simplification so that $P(t)$ is the only unknown here. Replace w with $-dm/dt$. The minus sign is included because the Bourdon element loses mass as time passes.

$$-\frac{dm}{dt} = \frac{\pi[P(t)-p_o]R^4\bar{\rho}}{8\mu L}$$

Replace m with the molar mass M times the number of moles n in the Bourdon element (a function of time as well).

$$-\frac{d(Mn)}{dt} = \frac{\pi[P(t) - p_o]R^4\bar{\rho}}{8\mu L}$$

M is assumed to be constant, so it can be pulled in front of the derivative.

$$-M\frac{dn}{dt} = \frac{\pi[P(t) - p_o]R^4\bar{\rho}}{8\mu L}$$

n can be written in terms of P using the equation of state,

$$PV = nR_gT \quad \rightarrow \quad n = \frac{PV}{R_gT}.$$

Substitute this into the Hagen-Poiseuille equation.

$$-M\frac{d}{dt}\left(\frac{PV}{R_gT}\right) = \frac{\pi[P(t) - p_o]R^4\bar{\rho}}{8\mu L}$$

V, the volume of the Bourdon element, is assumed to be constant, so it can be pulled in front of the derivative. R_g and T are also assumed to be constant.

$$-\frac{MV}{R_gT}\frac{dP}{dt} = \frac{\pi[P(t) - p_o]R^4\bar{\rho}}{8\mu L}$$

This is an inhomogeneous first-order differential equation that can be solved with an integrating factor. Put it into standard form first.

$$\frac{dP}{dt} + \frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}P(t) = \frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}p_o$$

The integrating factor I has the form,

$$I = \exp\left(\int^t \frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}\,ds\right) = \exp\left(\frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}t\right).$$

Multiply both sides of the differential equation by I.

$$\exp\left(\frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}t\right)\frac{dP}{dt} + \frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}\exp\left(\frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}t\right)P(t) = \frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}p_o\exp\left(\frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}t\right)$$

The left side is just $d/dt(IP)$ as a result of the product rule.

$$\frac{d}{dt}\left[\exp\left(\frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}t\right)P(t)\right] = \frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}p_o\exp\left(\frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}t\right)$$

Integrate both sides of the equation with respect to t.

$$\exp\left(\frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}t\right)P(t) = \int^t \frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}p_o\exp\left(\frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}s\right)\,ds + C_1$$

Evaluate the integral on the right side.

$$\exp\left(\frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}t\right)P(t) = p_o\exp\left(\frac{R_gT}{MV}\frac{\pi R^4\bar{\rho}}{8\mu L}t\right) + C_1$$

Divide both sides by I to solve for $P(t)$.

$$P(t) = p_o + C_1 \exp\left(-\frac{R_g T}{MV}\frac{\pi R^4 \bar{\rho}}{8\mu L}t\right)$$

The constant of integration is determined from the initial condition $P(0) = p_i$.

$$P(0) = p_o + C_1 = p_i \quad \rightarrow \quad C_1 = p_i - p_o$$

Thus, we have

$$P(t) = p_o + (p_i - p_o)\exp\left(-\frac{R_g T}{MV}\frac{\pi R^4 \bar{\rho}}{8\mu L}t\right)$$

for the pressure inside the Bourdon element as a function of time.

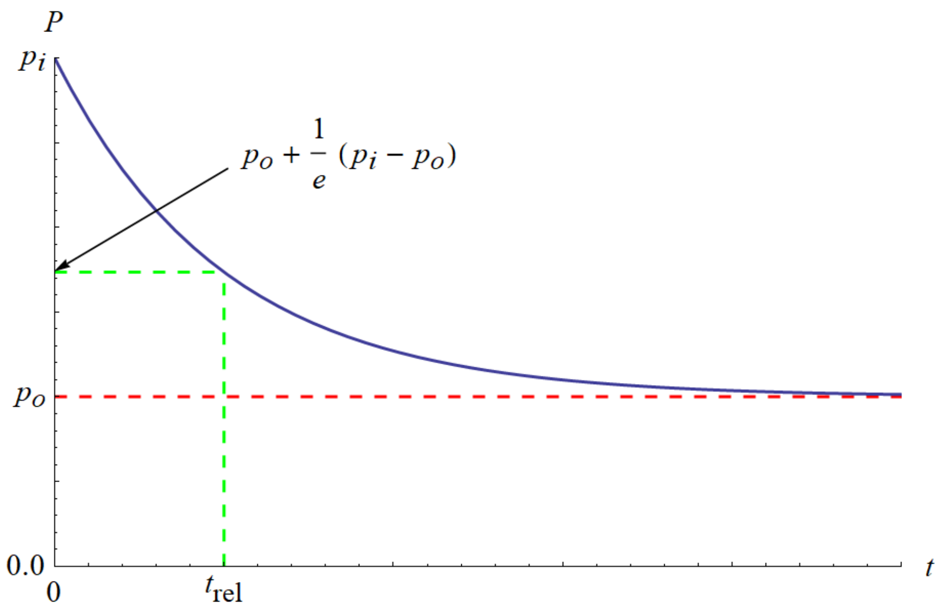

Figure 3: This is a plot of the pressure inside the Bourdon element as a function of time. Note that initially the pressure is p_i and that as t goes to infinity, it comes to equilibrium at p_o.

t_{rel}, the quantity we're interested in, is the time for the pressure to drop to $1/e$ of its initial value. This is found by setting the exponent of the exponential function equal to -1 and then solving for the time.

$$P(t_{\text{rel}}) = p_o + \frac{1}{e}(p_i - p_o) \quad \Rightarrow \quad -\frac{R_g T}{MV}\frac{\pi R^4 \bar{\rho}}{8\mu L}t_{\text{rel}} = -1$$

Solve this equation for t_{rel}.

$$\begin{aligned} t_{\text{rel}} &= \frac{8\mu L}{\pi R^4 \bar{\rho}}\frac{MV}{R_g T} \\ &= \frac{8\mu L}{\pi \left(\frac{D}{2}\right)^4 \bar{\rho}}\frac{MV}{R_g T} \\ &= \frac{128}{\pi}\frac{\mu V L}{D^4}\frac{M}{\bar{\rho} R_g T} \end{aligned}$$

The average density $\bar{\rho}$ can be related to the average pressure $\bar{p}$ using the equation of state.

$$\bar{p}V = nR_gT = \frac{m}{M}R_gT \quad \rightarrow \quad \bar{p} = \frac{m}{V}\frac{R_gT}{M} = \bar{\rho}\frac{R_gT}{M} \quad \rightarrow \quad \frac{1}{\bar{p}} = \frac{M}{\bar{\rho}R_gT}$$

Therefore,

$$t_{\text{rel}} = \frac{128}{\pi}\frac{\mu V L}{D^4\bar{p}},$$

where $\bar{p} = (p_i + p_o)/2$.

Part (d)

The approximations that were made (neglecting changes in air temperature, viscosity, density, and composition) are reasonable ones, provided that the plane does not fly too high. A small aircraft is not only expected to fly at low altitudes, but also to have a constant rate of climb. This sort of indicator would therefore be useful for one.

Part (e)

As the aircraft ascends (positive rate of climb), the pressure outside decreases and the Bourdon element uncoils, making the needle point to the right. Conversely, as the aircraft descends (negative rate of climb), the pressure outside increases and the Bourdon element coils up, making the needle point to the left.