Problem 2D.2

Drainage of liquids[9] (see Fig. 2D.2). How much liquid clings to the inside surface of a large vessel when it is drained? As shown in the figure there is a thin film of liquid left behind on the wall as the liquid level in the vessel falls. The local film thickness is a function of both z (the distance down from the initial liquid level) and t (the elapsed time).

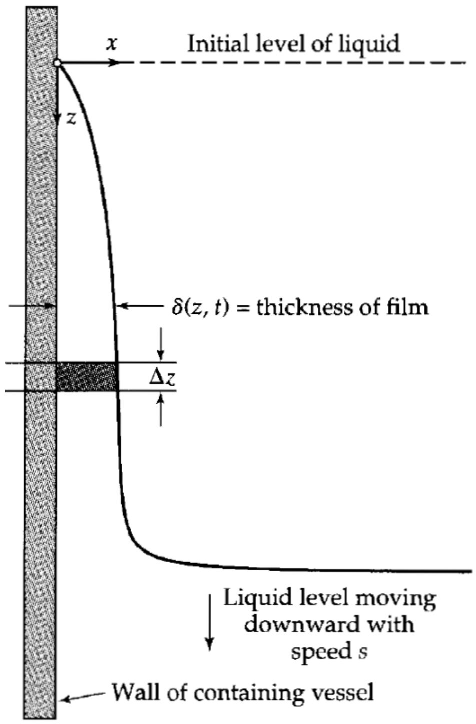

Fig. 2D.2 Clinging of a viscous fluid to wall of vessel during draining.

(a) Make an unsteady-state mass balance on a portion of the film between z and $z + \Delta z$ to get

$$\frac{\partial}{\partial z}\langle v_z \rangle \delta = -\frac{\partial \delta}{\partial t} \tag{2D.2-1}$$

(b) Use Eq. 2.2-18 and a quasi-steady-assumption to obtain the following first-order partial differential equation for $\delta(z,t)$:

$$\frac{\partial \delta}{\partial t} + \frac{\rho g}{\mu}\delta^2 \frac{\partial \delta}{\partial z} = 0 \tag{2D.2-2}$$

(c) Solve this equation to get

$$\delta(z,t) = \sqrt{\frac{\mu}{\rho g}\frac{z}{t}} \tag{2D.2-3}$$

What restrictions have to be placed on this result?

Solution

[9] J. J. van Rossum, *Appl. Sci. Research*, **47**, 121-144 (1958); see also V. G. Levich, *Physicochemical Hydrodynamics*, Prentice-Hall, Englewood Cliffs, N.J. (1962), Chapter 12.

Part (a)

The law of conservation of mass states that matter is neither created nor destroyed. The same amount of fluid that enters a shell per unit time must leave at that same rate; otherwise, fluid will build up or accumulate within the shell. The mathematical expression for this idea, a mass balance, is as follows.

$$\text{rate of mass in} - \text{rate of mass out} = \text{rate of mass accumulation}$$

Because of viscosity, the fluid in this problem doesn't flow freely. Some of it gets stuck to the

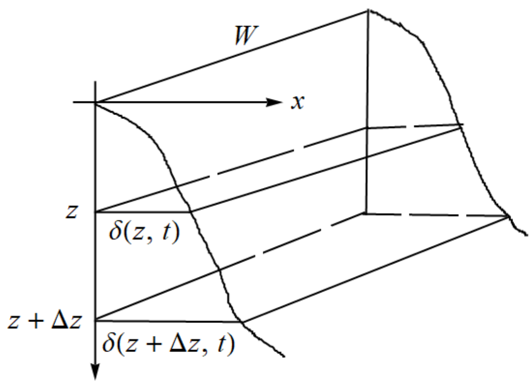

Figure 1: This is the shell over which the mass balance is made for liquid draining.

wall, so the rate of accumulation is not zero. Mass flows in at z and out at $z + \Delta z$, so with the shell in Figure 1 the mass balance becomes

$$\left.\frac{dm}{dt}\right|_z - \left.\frac{dm}{dt}\right|_{z+\Delta z} = \left.\frac{dm}{dt}\right|_{\text{shell}}.$$

Mass is density times volume.

$$\left.\frac{d(\rho V)}{dt}\right|_z - \left.\frac{d(\rho V)}{dt}\right|_{z+\Delta z} = \left.\frac{d(\rho V)}{dt}\right|_{\text{shell}}$$

As the density is assumed to be constant, it can be pulled out of each derivative and cancelled.

$$\left.\frac{dV}{dt}\right|_z - \left.\frac{dV}{dt}\right|_{z+\Delta z} = \left.\frac{dV}{dt}\right|_{\text{shell}}$$

For the left-hand side, the volumetric flow rate is the average velocity times the cross-sectional area that the fluid flows through.

$$(\langle v\rangle A)|_z - (\langle v\rangle A)|_{z+\Delta z} = \left.\frac{dV}{dt}\right|_{\text{shell}}$$

The cross-sectional areas at z and $z + \Delta z$ are $W\delta(z,t)$ and $W\delta(z + \Delta z, t)$, respectively.

$$\langle v\rangle|_z \left[W\delta(z,t)\right] - \langle v\rangle|_{z+\Delta z} \left[W\delta(z+\Delta z,t)\right] = \left.\frac{dV}{dt}\right|_{\text{shell}}$$

Factor $-W$ from the left side.

$$-W[\langle v\rangle|_{z+\Delta z}\,\delta(z+\Delta z,t)-\langle v\rangle|_z\,\delta(z,t)]=\left.\frac{dV}{dt}\right|_{\text{shell}}$$

For the right-hand side, an expression for the volume of the shell from z to Δz can be obtained by integrating the cross-sectional area over that interval.

$$V|_{\text{shell}}=\int_z^{z+\Delta z}A(s)\,ds$$

We get

$$-W[\langle v\rangle|_{z+\Delta z}\,\delta(z+\Delta z,t)-\langle v\rangle|_z\,\delta(z,t)]=\frac{d}{dt}\int_z^{z+\Delta z}[W\delta(s,t)]\,ds.$$

Now take the limit as $\Delta z\to 0$. It is assumed that δ is continuous and differentiable.

$$\lim_{\Delta z\to 0}-W[\langle v\rangle|_{z+\Delta z}\,\delta(z+\Delta z,t)-\langle v\rangle|_z\,\delta(z,t)]=\lim_{\Delta z\to 0}\frac{d}{dt}\int_z^{z+\Delta z}[W\delta(s,t)]\,ds$$

Bring the limit inside the derivative.

$$-W\lim_{\Delta z\to 0}[\langle v\rangle|_{z+\Delta z}\,\delta(z+\Delta z,t)-\langle v\rangle|_z\,\delta(z,t)]=\frac{d}{dt}\lim_{\Delta z\to 0}\int_z^{z+\Delta z}[W\delta(s,t)]\,ds$$

The interval of integration becomes so small that the integrand is essentially constant and can be pulled in front of the integral.

$$-W\lim_{\Delta z\to 0}[\langle v\rangle|_{z+\Delta z}\,\delta(z+\Delta z,t)-\langle v\rangle|_z\,\delta(z,t)]=\frac{d}{dt}\lim_{\Delta z\to 0}[W\delta(z,t)]\int_z^{z+\Delta z}ds$$

Evaluate the integral.

$$-W\lim_{\Delta z\to 0}[\langle v\rangle|_{z+\Delta z}\,\delta(z+\Delta z,t)-\langle v\rangle|_z\,\delta(z,t)]=\frac{d}{dt}\lim_{\Delta z\to 0}[W\delta(z,t)]\Delta z$$

W and Δz are constants with respect to time.

$$-W\lim_{\Delta z\to 0}[\langle v\rangle|_{z+\Delta z}\,\delta(z+\Delta z,t)-\langle v\rangle|_z\,\delta(z,t)]=W\frac{\partial\delta}{\partial t}\left(\lim_{\Delta z\to 0}\Delta z\right)$$

Divide both sides by $W\Delta z$.

$$-\lim_{\Delta z\to 0}\frac{\langle v\rangle|_{z+\Delta z}\,\delta(z+\Delta z,t)-\langle v\rangle|_z\,\delta(z,t)}{\Delta z}=\frac{\partial\delta}{\partial t}$$

The limit on the left is how the first derivative of $\langle v\rangle\delta$ with respect to z is defined.

$$-\frac{\partial}{\partial z}\langle v_z\rangle\delta=\frac{\partial\delta}{\partial t}$$

Therefore,

$$\frac{\partial}{\partial z}\langle v_z\rangle\delta=-\frac{\partial\delta}{\partial t}.$$

Part (b)

Eq. 2.2-18 gives the velocity distribution for a fluid flowing down an inclined plane.

$$\begin{aligned} v_z &= \frac{\rho g \delta^2 \cos\beta}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right] \qquad (2.2\text{-}18) \\ &= \frac{\rho g \cos\beta}{2\mu}(\delta^2 - x^2) \end{aligned}$$

To adapt the formula to this problem, set $\beta = 0$ because the wall is vertical. To obtain the average velocity $\langle v \rangle$, integrate this velocity over the cross-sectional area that the fluid flows through and then divide by that area.

$$\begin{aligned} \langle v \rangle &= \frac{1}{A}\int v_z\,dA \\ &= \frac{1}{W\delta(z,t)}\int_0^{\delta(z,t)} v_z(W\,dx) \\ &= \frac{1}{\delta(z,t)}\int_0^{\delta(z,t)} v_z\,dx \\ &= \frac{1}{\delta}\int_0^{\delta} \frac{\rho g}{2\mu}(\delta^2 - x^2)\,dx \\ &= \frac{\rho g}{2\mu\delta}\left(\delta^2 x - \frac{x^3}{3}\right)\Bigg|_0^{\delta} \\ &= \frac{\rho g}{2\mu\delta}\left(\frac{2\delta^3}{3}\right) \\ &= \frac{\rho g \delta^2}{3\mu} \end{aligned}$$

Substitute this formula for $\langle v \rangle$ into the mass balance.

$$\begin{aligned} \frac{\partial}{\partial z}\left(\frac{\rho g \delta^2}{3\mu}\right)\delta &= -\frac{\partial \delta}{\partial t} \\ \frac{\partial}{\partial z}\left(\frac{\rho g \delta^3}{3\mu}\right) &= -\frac{\partial \delta}{\partial t} \\ \frac{\rho g \delta^2}{\mu}\frac{\partial \delta}{\partial z} &= -\frac{\partial \delta}{\partial t} \end{aligned}$$

Therefore, we obtain a nonlinear first-order partial differential equation (PDE) for the film thickness.

$$\frac{\partial \delta}{\partial t} + \frac{\rho g}{\mu}\delta^2\frac{\partial \delta}{\partial z} = 0$$

Part (c)

For a function of two variables $\delta = \delta(z, t)$, its differential is defined as

$$d\delta = \frac{\partial \delta}{\partial z}\, dz + \frac{\partial \delta}{\partial t}\, dt.$$

Divide both sides by dt.

$$\frac{d\delta}{dt} = \frac{\partial \delta}{\partial z}\frac{dz}{dt} + \frac{\partial \delta}{\partial t}$$

This equation is marvelous because it tells us the relationship between an ordinary derivative and the partial derivatives of a function. Comparing the PDE we have with this equation, we see that along the curves in the tz-plane (known as characteristics) defined by

$$\frac{dz}{dt} = \frac{\rho g}{\mu}\delta^2, \tag{1}$$

the PDE simplifies to an ordinary differential equation (ODE),

$$\frac{d\delta}{dt} = 0. \tag{2}$$

Integrate both sides of equation (2) with respect to t to solve for δ.

$$\delta = f(\xi),$$

where f is an arbitrary function and ξ is a coordinate along the characteristics. ξ can be written in terms of z and t by solving equation (1). Equation (2) tells us that δ is a constant with respect to time along the characteristics, so equation (1) can be solved by integrating both sides with respect to t.

$$z = \frac{\rho g}{\mu}\delta^2 t + \xi$$

Solve for ξ.

$$\xi = z - \frac{\rho g}{\mu}\delta^2 t$$

Thus, the general solution for δ is

$$\delta(z, t) = f\left(z - \frac{\rho g}{\mu}\delta^2 t\right),$$

though this is of little interest to us. Since the origin of the coordinate system is chosen so that the film thickness starts to develop at $z = 0$, the solution on the $\xi = 0$ characteristic is the one we care about.

$$0 = z - \frac{\rho g}{\mu}\delta^2 t$$

Therefore,

$$\delta(z, t) = \sqrt{\frac{\mu}{\rho g}\frac{z}{t}}.$$

The solution only applies above the liquid level: $z > 0$, $t > 0$, and $z < st$. This can be written compactly as $0 < z < st$.

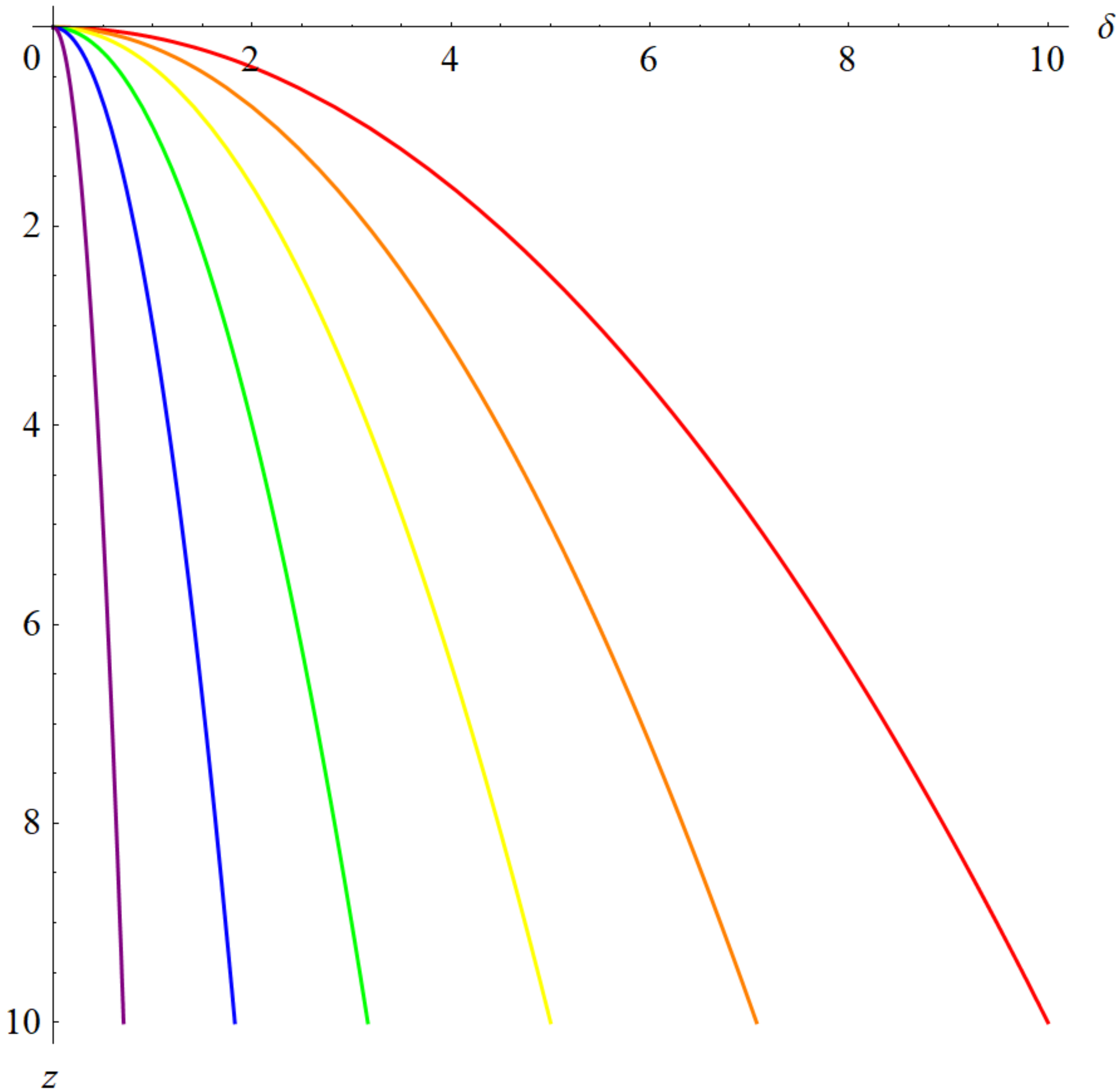

Figure 2: This is a plot of the film thickness δ vs. z at various times with $\mu/\rho g$ set equal to 1. The curves in red, orange, yellow, green, blue, and purple correspond to t equal to 0.1, 0.2, 0.4, 1, 3, and 20, respectively.