Problem 2B.10

Incompressible flow in a slightly tapered tube. An incompressible fluid flows through a tube of circular cross section, for which the tube radius changes linearly from $R_0$ at the tube entrance to a slightly smaller value $R_L$ at the tube exit. Assume that the Hagen-Poiseuille equation is approximately valid over a differential length, $dz$, of the tube so that the mass flow rate is

$$w = \frac{\pi [R(z)]^4 \rho}{8 \mu} \left( -\frac{d\mathcal{P}}{dz} \right)$$

This is a differential equation for $\mathcal{P}$ as a function of $z$, but, when the explicit expression for $R(z)$ is inserted, it is not easily solved.

(a) Write down the expression for $R$ as a function of $z$.

(b) Change the independent variable in Eq. 2B.10-1 to $R$, so that it becomes

$$w = \frac{\pi R^4 \rho}{8 \mu} \left( -\frac{d\mathcal{P}}{dR} \right) \left( \frac{R_L - R_0}{L} \right)$$

(c) Integrate the equation, and then show that the solution can be rearranged to give

$$w = \frac{\pi (\mathcal{P}_0 - \mathcal{P}_L) R_0^4 \rho}{8 \mu L} \left[ 1 - \frac{1 + (R_L/R_0) + (R_L/R_0)^2 - 3(R_L/R_0)^3}{1 + (R_L/R_0) + (R_L/R_0)^2} \right]$$

Interpret the result. The approximation used here that a flow between nonparallel surfaces can be regarded locally as flow between parallel surfaces is sometimes referred to as the lubrication approximation and is widely used in the theory of lubrication. By making a careful order-of-magnitude analysis, it can be shown that, for this problem, the lubrication approximation is valid as long as

$$\frac{R_L}{R_0} \left( 1 - \left( \frac{R_L}{R_0} \right)^2 \right)^2 \ll 1$$

Solution

Part (a)

$R$ needs to be a linear function of $z$ that is equal to $R_0$ at $z = 0$ and $R_L$ at $z = L$. It has the general form,

$$R(z) = mz + b.$$

Apply the conditions to determine the slope $m$ and slope-intercept $b$.

At $z = 0$: $R(0) = m \cdot 0 + b = R_0$

At $z = L$: $R(L) = m \cdot L + b = R_L$

Solve this system of equations for $m$ and $b$.

$$m = \frac{R_L - R_0}{L} \quad \text{and} \quad b = R_0$$

---


www.stemjock.com
Therefore, 
\[ R(z) = \frac{R_L - R_0}{L} z + R_0. \]

**Part (b)**

Now that \( R(z) \) is determined, the chain rule can be used to write the derivative of \( \mathcal{P} \) with respect to \( R \).

\[ \frac{d\mathcal{P}}{dz} = \frac{d\mathcal{P}}{dR} \frac{dR}{dz} \]

Take the first derivative of \( R(z) \) and plug it in for \( dR/dz \).

\[ \frac{d\mathcal{P}}{dz} = \frac{d\mathcal{P}}{dR} \left( \frac{R_L - R_0}{L} \right) \]

Substitute this result into Eq. 2B.10-1.

\[ w = \frac{\pi}{8\mu} \left[ \mathcal{P}_L - (R_L - R_0) \right] \]

Therefore,

\[ w = \frac{\pi R^4 \rho}{8\mu} \left( \frac{\mathcal{P}_0 - \mathcal{P}_L}{R_L - R_0} \right) \]

**Part (c)**

Solve the differential equation with the method of separation of variables.

\[ \frac{dR}{R^4 w} = \frac{\pi \rho}{8\mu} (-d\mathcal{P}) \left( \frac{R_L - R_0}{L} \right) \]

Integrate both sides.

\[ \int_{R_0}^{R_L} \frac{dR}{R^4} w = \int_{\mathcal{P}_L}^{\mathcal{P}_0} \frac{\pi \rho}{8\mu} (-d\mathcal{P}) \left( \frac{R_L - R_0}{L} \right) \]

Proceed with the integration.

\[ \frac{w}{3} \left[ \frac{1}{R^3} \right]_{R_0}^{R_L} = \frac{\pi \rho}{8\mu} (\mathcal{P}_L - \mathcal{P}_0) \left( \frac{R_L - R_0}{L} \right) \]

\[ \frac{w}{3} \left( \frac{1}{R^3_0} - \frac{1}{R^3_L} \right) = \frac{\pi \rho}{8\mu} (\mathcal{P}_0 - \mathcal{P}_L) \left( \frac{R_L - R_0}{L} \right) \]

\[ \frac{w}{3} \left( \frac{R^3_L - R^3_0}{R^3_0 R^3_L} \right) = \frac{\pi \rho}{8\mu} (\mathcal{P}_0 - \mathcal{P}_L) \left( \frac{R_L - R_0}{L} \right) \]

\[ \frac{w}{3} \left[ \frac{(R_L - R_0)(R^2_L + R_0 R_L + R^2_0)}{R^3_0 R^3_L} \right] = \frac{\pi \rho}{8\mu} (\mathcal{P}_0 - \mathcal{P}_L) \left( \frac{R_L - R_0}{L} \right) \]

Multiply both sides by \( R^4_0 \).

\[ \frac{w}{3} \left[ \frac{R_0(R^2_L + R_0 R_L + R^2_0)}{R^3_L} \right] = \frac{\pi (\mathcal{P}_0 - \mathcal{P}_L) R^4_0 \rho}{8\mu L} \]

www.stemjock.com
Expand the term in square brackets.

\[
\frac{w}{3} \left( \frac{R_0}{R_L} + \frac{R_0^2}{R_L^2} + \frac{R_0^3}{R_L^3} \right) = \frac{\pi (\mathcal{P}_0 - \mathcal{P}_L) R_0^4 \rho}{8 \mu L}
\]

Solve this equation for \( w \) now.

\[
w = \frac{\pi (\mathcal{P}_0 - \mathcal{P}_L) R_0^4 \rho}{8 \mu L} \left( \frac{3}{\frac{R_0}{R_L} + \frac{R_0^2}{R_L^2} + \frac{R_0^3}{R_L^3}} \right)
\]

\[
= \frac{\pi (\mathcal{P}_0 - \mathcal{P}_L) R_0^4 \rho}{8 \mu L} \left[ \frac{3}{\frac{R_0^3}{R_L^3} \left( \frac{R_0^2}{R_L^2} + \frac{R_0}{R_L} + 1 \right)} \right]
\]

\[
= \frac{\pi (\mathcal{P}_0 - \mathcal{P}_L) R_0^4 \rho}{8 \mu L} \left( \frac{3 \frac{R_0^3}{R_L^3}}{\frac{R_0^3}{R_L^3} + \frac{R_0}{R_L} + 1} \right)
\]

\[
= \frac{\pi (\mathcal{P}_0 - \mathcal{P}_L) R_0^4 \rho}{8 \mu L} \left( \frac{1 + \frac{R_L}{R_0} + \frac{R_0^2}{R_L^2} - 1 - \frac{R_L}{R_0} - \frac{R_0^2}{R_L^2} + 3 \frac{R_0^3}{R_L^3}}{1 + \frac{R_L}{R_0} + \frac{R_0^2}{R_L^2}} \right)
\]

\[
= \frac{\pi (\mathcal{P}_0 - \mathcal{P}_L) R_0^4 \rho}{8 \mu L} \left( \frac{1 + \frac{R_L}{R_0} + \frac{R_0^2}{R_L^2} - \frac{R_L}{R_0} - \frac{R_0^2}{R_L^2} + 3 \frac{R_0^3}{R_L^3}}{1 + \frac{R_L}{R_0} + \frac{R_0^2}{R_L^2}} \right)
\]

\[
= \frac{\pi (\mathcal{P}_0 - \mathcal{P}_L) R_0^4 \rho}{8 \mu L} \left( \frac{1 - \frac{1 + \frac{R_L}{R_0} + \frac{R_0^2}{R_L^2} - 3 \frac{R_0^3}{R_L^3}}{1 + \frac{R_L}{R_0} + \frac{R_0^2}{R_L^2}}}{1 + \frac{R_L}{R_0} + \frac{R_0^2}{R_L^2}} \right)
\]

Therefore,

\[
w = \frac{\pi (\mathcal{P}_0 - \mathcal{P}_L) R_0^4 \rho}{8 \mu L} \left[ 1 - \frac{1 + \left( \frac{R_L}{R_0} \right) + \left( \frac{R_L}{R_0} \right)^2 - 3 \left( \frac{R_L}{R_0} \right)^3}{1 + \left( \frac{R_L}{R_0} \right) + \left( \frac{R_L}{R_0} \right)^2} \right].
\]

In the event that \( R_0 = R_L \), there is no taper in the tube, and the formula for \( w \) simplifies to the Hagen-Poiseuille equation. If \( R_0 \gg R_L \), then the mass flow rate tends to 0. This doesn’t make sense, though, because as long as \( R_L \) is big enough, something should flow through the tube exit regardless of how big \( R_0 \) is. In the problem statement the assumption was made that \( R_L \) was only slightly less than \( R_0 \), so \( R_0 \gg R_L \) is not a case we need to consider.