Problem 2B.10

Incompressible flow in a slightly tapered tube. An incompressible fluid flows through a tube of circular cross section, for which the tube radius changes linearly from R_0 at the tube entrance to a slightly smaller value R_L at the tube exit. Assume that the Hagen-Poiseuille equation is *approximately* valid over a differential length, dz, of the tube so that the mass flow rate is

$$w = \frac{\pi[R(z)]^4\rho}{8\mu}\left(-\frac{d\mathscr{P}}{dz}\right) \tag{2B.10-1}$$

This is a differential equation for $\mathscr{P}$ as a function of z, but, when the explicit expression for $R(z)$ is inserted, it is not easily solved.

(a) Write down the expression for R as a function of z.

(b) Change the independent variable in Eq. 2B.10-1 to R, so that it becomes

$$w = \frac{\pi R^4\rho}{8\mu}\left(-\frac{d\mathscr{P}}{dR}\right)\left(\frac{R_L - R_0}{L}\right) \tag{2B.10-2}$$

(c) Integrate the equation, and then show that the solution can be rearranged to give

$$w = \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R_0^4\rho}{8\mu L}\left[1 - \frac{1 + (R_L/R_0) + (R_L/R_0)^2 - 3(R_L/R_0)^3}{1 + (R_L/R_0) + (R_L/R_0)^2}\right] \tag{2B.10-3}$$

Interpret the result. The approximation used here that a flow between nonparallel surfaces can be regarded locally as flow between parallel surfaces is sometimes referred to as the *lubrication approximation* and is widely used in the theory of lubrication. By making a careful order-of-magnitude analysis, it can be shown that, for this problem, the lubrication approximation is valid as long as[4]

$$\frac{R_L}{R_0}\left(1 - \left(\frac{R_L}{R_0}\right)^2\right) \ll 1 \tag{2B.10-4}$$

Solution

Part (a)

R needs to be a linear function of z that is equal to R_0 at $z = 0$ and R_L at $z = L$. It has the general form,

$$R(z) = mz + b.$$

Apply the conditions to determine the slope m and slope-intercept b.

$$\begin{aligned} \text{At } z = 0: \quad & R(0) = m \cdot 0 + b = R_0 \\ \text{At } z = L: \quad & R(L) = m \cdot L + b = R_L \end{aligned}$$

Solve this system of equations for m and b.

$$m = \frac{R_L - R_0}{L} \quad \text{and} \quad b = R_0$$

[4] R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1, Wiley-Interscience, New York, 2nd edition (1987), pp. 16-18.

Therefore,

$$R(z) = \frac{R_L - R_0}{L} z + R_0.$$

Part (b)

Now that $R(z)$ is determined, the chain rule can be used to write the derivative of $\mathscr{P}$ with respect to R.

$$\frac{d\mathscr{P}}{dz} = \frac{d\mathscr{P}}{dR}\frac{dR}{dz}$$

Take the first derivative of $R(z)$ and plug it in for dR/dz.

$$\frac{d\mathscr{P}}{dz} = \frac{d\mathscr{P}}{dR}\left(\frac{R_L - R_0}{L}\right)$$

Substitute this result into Eq. 2B.10-1.

$$\begin{aligned} w &= \frac{\pi [R(z)]^4 \rho}{8\mu}\left(-\frac{d\mathscr{P}}{dz}\right) \\ &= \frac{\pi R^4 \rho}{8\mu}\left[-\frac{d\mathscr{P}}{dR}\left(\frac{R_L - R_0}{L}\right)\right] \end{aligned}$$

Therefore,

$$w = \frac{\pi R^4 \rho}{8\mu}\left(-\frac{d\mathscr{P}}{dR}\right)\left(\frac{R_L - R_0}{L}\right).$$

Part (c)

Solve the differential equation with the method of separation of variables.

$$\frac{dR}{R^4} w = \frac{\pi\rho}{8\mu}(-d\mathscr{P})\left(\frac{R_L - R_0}{L}\right)$$

Integrate both sides.

$$\int_{R_0}^{R_L} \frac{dR}{R^4} w = \int_{\mathscr{P}_0}^{\mathscr{P}_L} \frac{\pi\rho}{8\mu}(-d\mathscr{P})\left(\frac{R_L - R_0}{L}\right)$$

Proceed with the integration.

$$\begin{aligned} -\frac{w}{3}\frac{1}{R^3}\bigg|_{R_0}^{R_L} &= -\frac{\pi\rho}{8\mu}(\mathscr{P}_L - \mathscr{P}_0)\left(\frac{R_L - R_0}{L}\right) \\ \frac{w}{3}\left(\frac{1}{R_0^3} - \frac{1}{R_L^3}\right) &= \frac{\pi\rho}{8\mu}(\mathscr{P}_0 - \mathscr{P}_L)\left(\frac{R_L - R_0}{L}\right) \\ \frac{w}{3}\left(\frac{R_L^3 - R_0^3}{R_0^3 R_L^3}\right) &= \frac{\pi\rho}{8\mu}(\mathscr{P}_0 - \mathscr{P}_L)\left(\frac{R_L - R_0}{L}\right) \\ \frac{w}{3}\left[\frac{\cancel{(R_L - R_0)}(R_L^2 + R_0 R_L + R_0^2)}{R_0^3 R_L^3}\right] &= \frac{\pi\rho}{8\mu}(\mathscr{P}_0 - \mathscr{P}_L)\left(\frac{\cancel{R_L - R_0}}{L}\right) \end{aligned}$$

Multiply both sides by R_0^4.

$$\frac{w}{3}\left[\frac{R_0(R_L^2 + R_0 R_L + R_0^2)}{R_L^3}\right] = \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R_0^4\rho}{8\mu L}$$

Expand the term in square brackets.

$$\frac{w}{3}\left(\frac{R_0}{R_L}+\frac{R_0^2}{R_L^2}+\frac{R_0^3}{R_L^3}\right)=\frac{\pi(\mathscr{P}_0-\mathscr{P}_L)R_0^4\rho}{8\mu L}$$

Solve this equation for w now.

$$\begin{aligned}
w &= \frac{\pi(\mathscr{P}_0-\mathscr{P}_L)R_0^4\rho}{8\mu L}\left(\frac{3}{\frac{R_0}{R_L}+\frac{R_0^2}{R_L^2}+\frac{R_0^3}{R_L^3}}\right) \\
&= \frac{\pi(\mathscr{P}_0-\mathscr{P}_L)R_0^4\rho}{8\mu L}\left[\frac{3}{\frac{R_0^3}{R_L^3}\left(\frac{R_L^2}{R_0^2}+\frac{R_L}{R_0}+1\right)}\right] \\
&= \frac{\pi(\mathscr{P}_0-\mathscr{P}_L)R_0^4\rho}{8\mu L}\left(\frac{3\frac{R_L^3}{R_0^3}}{\frac{R_L^2}{R_0^2}+\frac{R_L}{R_0}+1}\right) \\
&= \frac{\pi(\mathscr{P}_0-\mathscr{P}_L)R_0^4\rho}{8\mu L}\left(\frac{1+\frac{R_L}{R_0}+\frac{R_L^2}{R_0^2}-1-\frac{R_L}{R_0}-\frac{R_L^2}{R_0^2}+3\frac{R_L^3}{R_0^3}}{1+\frac{R_L}{R_0}+\frac{R_L^2}{R_0^2}}\right) \\
&= \frac{\pi(\mathscr{P}_0-\mathscr{P}_L)R_0^4\rho}{8\mu L}\left(\frac{1+\frac{R_L}{R_0}+\frac{R_L^2}{R_0^2}}{1+\frac{R_L}{R_0}+\frac{R_L^2}{R_0^2}}+\frac{-1-\frac{R_L}{R_0}-\frac{R_L^2}{R_0^2}+3\frac{R_L^3}{R_0^3}}{1+\frac{R_L}{R_0}+\frac{R_L^2}{R_0^2}}\right) \\
&= \frac{\pi(\mathscr{P}_0-\mathscr{P}_L)R_0^4\rho}{8\mu L}\left(1-\frac{1+\frac{R_L}{R_0}+\frac{R_L^2}{R_0^2}-3\frac{R_L^3}{R_0^3}}{1+\frac{R_L}{R_0}+\frac{R_L^2}{R_0^2}}\right)
\end{aligned}$$

Therefore,

$$w=\frac{\pi(\mathscr{P}_0-\mathscr{P}_L)R_0^4\rho}{8\mu L}\left[1-\frac{1+(R_L/R_0)+(R_L/R_0)^2-3(R_L/R_0)^3}{1+(R_L/R_0)+(R_L/R_0)^2}\right].$$

In the event that $R_0=R_L$, there is no taper in the tube, and the formula for w simplifies to the Hagen-Poiseuille equation. If $R_0 \gg R_L$, then the mass flow rate tends to 0. This doesn't make sense, though, because as long as R_L is big enough, something should flow through the tube exit regardless of how big R_0 is. In the problem statement the assumption was made that R_L was only slightly less than R_0, so $R_0 \gg R_L$ is not a case we need to consider.