# Problem 2B.10

**Incompressible flow in a slightly tapered tube.** An incompressible fluid flows through a tube of circular cross section, for which the tube radius changes linearly from $R_0$ at the tube entrance to a slightly smaller value $R_L$ at the tube exit. Assume that the Hagen-Poiseuille equation is *approximately* valid over a differential length, $dz$, of the tube so that the mass flow rate is

$$w = \frac{\pi[R(z)]^4\rho}{8\mu}\left(-\frac{d\mathscr{P}}{dz}\right) \tag{2B.10-1}$$

This is a differential equation for $\mathscr{P}$ as a function of $z$, but, when the explicit expression for $R(z)$ is inserted, it is not easily solved.

(a) Write down the expression for $R$ as a function of $z$.

(b) Change the independent variable in Eq. 2B.10-1 to $R$, so that it becomes

$$w = \frac{\pi R^4\rho}{8\mu}\left(-\frac{d\mathscr{P}}{dR}\right)\left(\frac{R_L - R_0}{L}\right) \tag{2B.10-2}$$

(c) Integrate the equation, and then show that the solution can be rearranged to give

$$w = \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R_0^4\rho}{8\mu L}\left[1 - \frac{1 + (R_L/R_0) + (R_L/R_0)^2 - 3(R_L/R_0)^3}{1 + (R_L/R_0) + (R_L/R_0)^2}\right] \tag{2B.10-3}$$

Interpret the result. The approximation used here that a flow between nonparallel surfaces can be regarded locally as flow between parallel surfaces is sometimes referred to as the *lubrication approximation* and is widely used in the theory of lubrication. By making a careful order-of-magnitude analysis, it can be shown that, for this problem, the lubrication approximation is valid as long as[4]

$$\frac{R_L}{R_0}\left(1 - \left(\frac{R_L}{R_0}\right)^2\right) \ll 1 \tag{2B.10-4}$$

[4] R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1, Wiley-Interscience, New York, 2nd edition (1987), pp. 16-18.