Problem 2B.11

The cone-and-plate viscometer (see Fig. 2B.11). A cone-and-plate viscometer consists of a flat plate and an inverted cone, whose apex just contacts the plate. The liquid whose viscosity is to be measured is placed in the gap between the cone and plate. The cone is rotated at a known angular velocity $\Omega$, and the torque $T_z$ required to turn the cone is measured. Find an expression for the viscosity of the fluid in terms of $\Omega$, $T_z$, and the angle $\psi_0$ between the cone and the plate. For commercial instruments $\psi_0$ is about 1 degree.

(a) Assume that locally the velocity distribution in the gap can be very closely approximated by that for flow between parallel plates, the upper one moving with a constant speed. Verify that this leads to the approximate velocity distribution (in spherical coordinates)

$$\frac{v_\phi}{r} = \Omega \left( \frac{\pi/2 - \theta}{\psi_0} \right)$$

(2B.11-1)

This approximation should be rather good, because $\psi_0$ is so small.

(b) From the velocity distribution in Eq. 2B.11-1 and Appendix B.1, show that a reasonable expression for the shear stress is

$$\tau_{\theta\phi} = \mu (\Omega / \psi_0)$$

(2B.11-2)

This result shows that the shear stress is uniform throughout the gap. It is this fact that makes the cone-and-plate viscometer quite attractive. The instrument is widely used, particularly in the polymer industry.

(c) Show that the torque required to turn the cone is given by

$$T_z = \frac{2}{3} \pi \mu \Omega R^3 / \psi_0$$

(2B.11-3)

This is the standard formula for calculating the viscosity from measurements of the torque and angular velocity for a cone-plate assembly with known $R$ and $\psi_0$.

(d) For a cone-and-plate instrument with radius 10 cm and angle $\psi_0$ equal to 0.5 degree, what torque (in dyn·cm) is required to turn the cone at an angular velocity of 10 radians per minute if the fluid viscosity is 100 cp?
Answer: (d) 40,000 dyn · cm

Solution

Part (a)

Because the linear velocity of the cone’s top varies with \( r \) \((V = \Omega r)\), we can treat each slice of the circle with circumference \( 2\pi r \) and thickness \( \Delta r \) as a linear element with length \( L = 2\pi r \) and thickness \( \Delta r \) moving at speed \( V \). The problem essentially boils down to Couette flow, where a fluid flows in a slit and one wall moves while the other remains stationary. This is illustrated in part (c) of Fig. 2B.11. Thus, a rectangular shell will be considered to determine the velocity distribution. We assume that the fluid flows in the \( x \)-direction and that its velocity varies in the \( y \)-direction.

\[ v_x = v_x(y) \]

Consequently, only \( \phi_{yx} \) (the \( x \)-momentum in the positive \( y \)-direction) and \( \phi_{xx} \) (the \( x \)-momentum in the positive \( x \)-direction) contribute to the momentum balance. We also assume that the flow is only due to the spinning top and is not a result of gravity or a pressure difference.

![Diagram showing the shell over which the momentum balance is made for the flow in a slit.](image)

Figure 1: This is the shell over which the momentum balance is made for the flow in a slit.

- Rate of \( x \)-momentum into the shell at \( x = 0 \): \((\Delta r \Delta y)\phi_{xx}|_{x=0}\)
- Rate of \( x \)-momentum into the shell at \( x = L \): \((\Delta r \Delta y)\phi_{xx}|_{x=L}\)
- Rate of \( x \)-momentum into the shell at \( y \): \((L\Delta r)\phi_{yx}|_{y}\)
- Rate of \( x \)-momentum into the shell at \( y + \Delta y \): \((L\Delta r)\phi_{yx}|_{y+\Delta y}\)
- Component of gravitational force on the shell in \( x \)-direction: 0
If we assume steady flow, then the momentum balance is

\[
\text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = 0.
\]

Considering only the \(x\)-component, we have

\[
(\Delta r \Delta y)|_{x=0} - (\Delta r \Delta y)|_{x=L} + (L \Delta r) \phi_{yx} |_{y} - (L \Delta r) \phi_{yx} |_{y+\Delta y} = 0.
\]

Factor the left side.

\[
-\Delta r \Delta y(\phi_{xx} |_{x=L} - \phi_{xx} |_{x=0}) - L \Delta r(\phi_{yx} |_{y} - \phi_{yx} |_{y}) = 0
\]

Divide both sides by \(L \Delta r \Delta y\).

\[
-\frac{\phi_{xx} |_{x=L} - \phi_{xx} |_{x=0}}{L} - \frac{\phi_{yx} |_{y} - \phi_{yx} |_{y}}{\Delta y} = 0
\]

Take the limit as \(\Delta y \to 0\).

\[
-\lim_{\Delta y \to 0} \frac{\phi_{xx} |_{x=L} - \phi_{xx} |_{x=0}}{L} - \frac{\phi_{yx} |_{y} - \phi_{yx} |_{y}}{\Delta y} = 0
\]

The second term is the first derivative of \(\phi_{yx}\) with respect to \(y\).

\[
-\frac{\phi_{xx} |_{x=L} - \phi_{xx} |_{x=0}}{L} - \frac{d\phi_{yx}}{dy} = 0
\]

Now substitute the expressions for \(\phi_{yx}\) and \(\phi_{xx}\).

\[
\phi_{yx} = \tau_{yx} + \rho v_{y} v_{x} = \tau_{yx}
\]

\[
\phi_{xx} = p \delta_{xx} + \tau_{xx} + \rho v_{x} v_{x} = \rho v_{x}^{2}
\]

Since \(v_{x}\) does not depend on \(x\), the \(\rho v_{x}^{2}\) terms cancel and we get

\[
-\frac{\phi_{xx} |_{x=L} - \phi_{xx} |_{x=0}}{L} - \frac{d\tau_{yx}}{dy} = 0
\]

or

\[
\frac{d\tau_{yx}}{dy} = 0.
\]

From Newton’s law of viscosity, \(\tau_{yx} = -\mu (dv_{x}/dy)\), so we have

\[
\frac{d}{dy} \left( -\mu \frac{dv_{x}}{dy} \right) = 0.
\]

Viscosity is assumed to be constant, so we can pull it and the minus sign in front of the derivative. Divide both sides by \(-\mu\).

\[
\frac{d^{2}v_{x}}{dy^{2}} = 0
\]

Using the coordinate system in part (c) of Fig. 2B.11, the boundary conditions are as follows.

B.C. 1: \(v_{x} = 0\) when \(y = 0\)

B.C. 2: \(v_{x} = V\) when \(y = b\)

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Integrate the differential equation with respect to \( y \).

\[
\frac{dv_x}{dy} = C_1
\]

Integrate the differential equation with respect to \( y \) once more.

\[
v_x(y) = C_1 y + C_2
\]

Apply the two boundary conditions now.

\[
v_x(0) = C_1(0) + C_2 = 0 \\
v_x(b) = C_1(b) + C_2 = V
\]

Solving this system of equations, we get \( C_1 = V/b \) and \( C_2 = 0 \). Hence, we have for the velocity distribution

\[
v_x(y) = \frac{V}{b} y.
\]

Now we can change back to rotational flow in spherical coordinates; that is, the fluid actually flows in the \( \phi \) direction and its velocity varies with \( r \) \([v_\phi = v_\phi(r)]\). We have \( V = \Omega r \), \( b = r \sin \psi_0 \approx r \psi_0 \), and \( y \approx r(\pi/2 - \theta) \).

\[
v_\phi = \frac{\Omega \psi_0 r}{r \psi_0} \left( \frac{\pi}{2} - \theta \right)
\]

Therefore,

\[
\frac{v_\phi}{r} = \Omega \left( \frac{(\pi/2) - \theta}{\psi_0} \right).
\]

**Part (b)**

The shear stress on a surface element of the cone by the fluid is given by \( \tau_{\theta \phi} \) (the \( \theta \)-direction is perpendicular to the surface element, and the shear is in the \( \phi \)-direction). According to Appendix B.1 on page 844,

\[
\tau_{\theta \phi} = -\mu \left[ \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right].
\]

Since the velocity is in the \( \phi \)-direction only, the second term in the square brackets is zero. Also, \( \theta \) is a constant on the cone’s surface \((\theta = \theta_0)\), so the formula for \( \tau_{\theta \phi} \) simplifies considerably.

\[
\tau_{\theta \phi} = -\mu \frac{\sin \theta_0}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta_0} \right)
\]

\[
= -\mu \frac{1}{r} \frac{\partial v_\phi}{\partial \theta}
\]

\[
= -\mu \frac{\partial}{\partial \theta} \left[ \Omega r \left( \frac{\pi}{2} - \theta \right) \right]
\]

\[
= -\mu \frac{\Omega r}{\psi_0} \left( -\frac{1}{\psi_0} \right)
\]

Therefore,

\[
\tau_{\theta \phi} = \mu \frac{\Omega}{\psi_0}.
\]

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**Part (c)**

Since the shearing force is in the $\phi$-direction, it is perpendicular to the radial direction. The torque then is just the product of this force with the moment arm.

$$T_z = rF$$

To get the shearing force, we multiply (integrate) the shear stress by the area of the cone that the fluid is in contact with.

$$T_z = \int r \cdot \tau_{\theta\phi}|_{\theta=\frac{\pi}{2} - \psi_0} \, dA$$

Since the fluid is at a lower value of $\theta$ than the cone’s surface at $\theta = \frac{\pi}{2} - \psi_0$, no minus sign is needed in front of $\tau_{\theta\phi}$. Looking at part (b) in Fig. 2B.11, we see that the area differential is $dA = r \, dr \, d\phi$.

$$T_z = \int_0^{2\pi} \int_0^R r \cdot \tau_{\theta\phi}|_{\theta=\frac{\pi}{2} - \psi_0} \, r \, dr \, d\phi$$

$$= \int_0^{2\pi} \int_0^R r^2 \left( \frac{\Omega}{\psi_0} \right) \, dr \, d\phi$$

$$= \frac{\Omega}{\psi_0} \left( \int_0^{2\pi} d\phi \right) \left( \int_0^R r^2 \, dr \right)$$

$$= \frac{\Omega}{\psi_0} (2\pi) \left( \frac{R^3}{3} \right)$$

Therefore,

$$T_z = \frac{2}{3} \pi \mu \Omega \frac{R^3}{\psi_0}.$$

**Part (d)**

We have the following values for the variables. The conversion factor for Pa to dyn/cm$^2$ is found in Table F.3-2 on page 869.

- $R = 10$ cm
- $\psi_0 = 0.5^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{\pi}{360} \text{ rad}$
- $\Omega = 10 \text{ rad/min} \times \frac{1 \text{ min}}{60 \text{ s}} = \frac{1}{6} \text{ rad/s}$
- $\mu = 100 \text{ cp} \times \frac{10^{-3} \text{ Pa} \cdot \text{s}}{1 \text{ cp}} = 1 \frac{\text{ dyn}}{\text{ cm}^2 \cdot \text{s}}$

Plugging the numbers into the formula for torque $T_z$, we get

$$T_z = \frac{2}{3} \pi \mu \Omega \frac{R^3}{\psi_0}$$

$$= \frac{2}{3} \pi \left( 1 \frac{\text{ dyn}}{\text{ cm}^2 \cdot \text{s}} \right) \left( \frac{1}{6} \frac{\text{ rad}}{\text{s}} \right) \frac{1}{\frac{\pi}{360} \text{ rad}}$$

$$= 40,000 \frac{\text{ dyn}}{\text{ cm}^2} \cdot \text{ cm}^3 = 40,000 \text{ dyn} \cdot \text{ cm}.$$