Problem 2B.12

Flow of a fluid in a network of tubes (see Fig. 2B.12). A fluid is flowing in laminar flow from A to B through a network of tubes, as depicted in the figure. Obtain an expression for the mass flow rate w of the fluid entering at A (or leaving at B) as a function of the modified pressure drop $\mathscr{P}_A - \mathscr{P}_B$. Neglect the disturbances at the various tube junctions.

$$\textit{Answer: } w = \frac{3\pi(\mathscr{P}_A - \mathscr{P}_B)R^4\rho}{20\mu L}$$

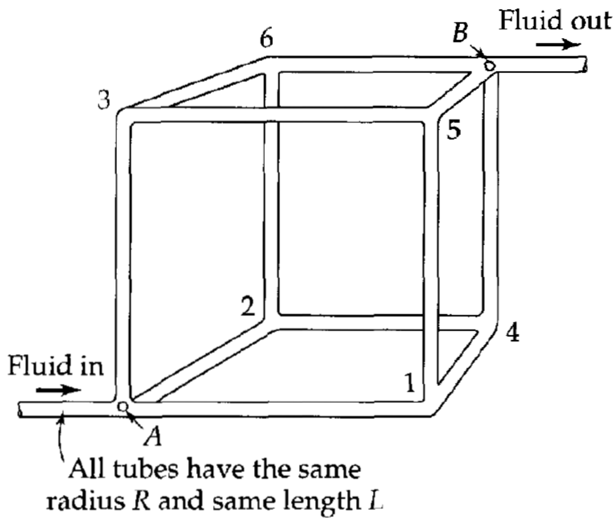

Fig. 2B.12 Flow of a fluid in a network with branching.

Solution

The main idea with this problem is the law of conservation of mass; that is, mass is neither created nor destroyed. Each of the corners in this network can be referred to as a node. The rate of mass flowing into a node must be equal to the rate of mass flowing out of it, assuming no mass accumulates there. Let w_A be the rate of mass flowing into node A and let w_B be the rate of mass flowing out of node B. In addition, let w_{ij} be the rate of mass flowing from node i to node j. As each tube is a cylinder with radius R and length L, the Hagen-Poiseuille equation can be used for the mass flow rate,

$$w_{ij} = \frac{\pi(\mathscr{P}_i - \mathscr{P}_j)R^4\rho}{8\mu L},$$

where $\mathscr{P}_i$ is the modified pressure at node i and $\mathscr{P}_j$ is the modified pressure at node j.

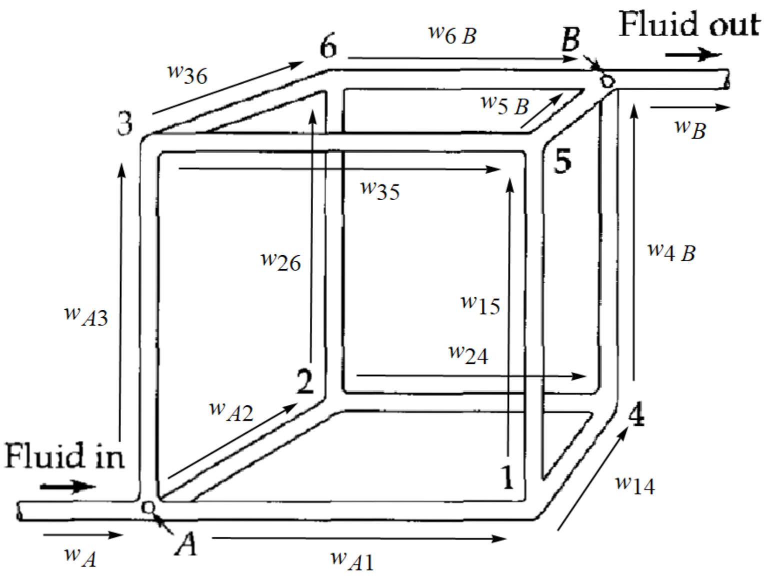

Figure 1: The network with rates of mass flow w_{ij} labeled for each tube. The assumed direction that the mass is flowing in each tube is shown as well.

Applying the law of conservation of mass to each node, we obtain the following eight equations.

$$\begin{aligned}
&\text{At node } A : w_A = w_{A1} + w_{A2} + w_{A3} \\
&\text{At node } 1 : w_{A1} = w_{14} + w_{15} \\
&\text{At node } 2 : w_{A2} = w_{24} + w_{26} \\
&\text{At node } 3 : w_{A3} = w_{35} + w_{36} \\
&\text{At node } 4 : w_{14} + w_{24} = w_{4B} \\
&\text{At node } 5 : w_{15} + w_{35} = w_{5B} \\
&\text{At node } 6 : w_{26} + w_{36} = w_{6B} \\
&\text{At node } B : w_{4B} + w_{5B} + w_{6B} = w_B
\end{aligned}$$

Substitute the Hagen-Poiseuille equation into the equations from nodes 1-6.

$$\begin{aligned}
w_{A1} = w_{14} + w_{15} \quad &\rightarrow \quad \frac{\pi(\mathscr{P}_A - \mathscr{P}_1)R^4\rho}{8\mu L} = \frac{\pi(\mathscr{P}_1 - \mathscr{P}_4)R^4\rho}{8\mu L} + \frac{\pi(\mathscr{P}_1 - \mathscr{P}_5)R^4\rho}{8\mu L} \\
w_{A2} = w_{24} + w_{26} \quad &\rightarrow \quad \frac{\pi(\mathscr{P}_A - \mathscr{P}_2)R^4\rho}{8\mu L} = \frac{\pi(\mathscr{P}_2 - \mathscr{P}_4)R^4\rho}{8\mu L} + \frac{\pi(\mathscr{P}_2 - \mathscr{P}_6)R^4\rho}{8\mu L} \\
w_{A3} = w_{35} + w_{36} \quad &\rightarrow \quad \frac{\pi(\mathscr{P}_A - \mathscr{P}_3)R^4\rho}{8\mu L} = \frac{\pi(\mathscr{P}_3 - \mathscr{P}_5)R^4\rho}{8\mu L} + \frac{\pi(\mathscr{P}_3 - \mathscr{P}_6)R^4\rho}{8\mu L} \\
w_{14} + w_{24} = w_{4B} \quad &\rightarrow \quad \frac{\pi(\mathscr{P}_1 - \mathscr{P}_4)R^4\rho}{8\mu L} + \frac{\pi(\mathscr{P}_2 - \mathscr{P}_4)R^4\rho}{8\mu L} = \frac{\pi(\mathscr{P}_4 - \mathscr{P}_B)R^4\rho}{8\mu L} \\
w_{15} + w_{35} = w_{5B} \quad &\rightarrow \quad \frac{\pi(\mathscr{P}_1 - \mathscr{P}_5)R^4\rho}{8\mu L} + \frac{\pi(\mathscr{P}_3 - \mathscr{P}_5)R^4\rho}{8\mu L} = \frac{\pi(\mathscr{P}_5 - \mathscr{P}_B)R^4\rho}{8\mu L} \\
w_{26} + w_{36} = w_{6B} \quad &\rightarrow \quad \frac{\pi(\mathscr{P}_2 - \mathscr{P}_6)R^4\rho}{8\mu L} + \frac{\pi(\mathscr{P}_3 - \mathscr{P}_6)R^4\rho}{8\mu L} = \frac{\pi(\mathscr{P}_6 - \mathscr{P}_B)R^4\rho}{8\mu L}
\end{aligned}$$

Multiply both sides of these equations by $8\mu L/\pi R^4\rho$.

$$
\begin{aligned}
(\mathscr{P}_A - \mathscr{P}_1) &= (\mathscr{P}_1 - \mathscr{P}_4) + (\mathscr{P}_1 - \mathscr{P}_5) \\
(\mathscr{P}_A - \mathscr{P}_2) &= (\mathscr{P}_2 - \mathscr{P}_4) + (\mathscr{P}_2 - \mathscr{P}_6) \\
(\mathscr{P}_A - \mathscr{P}_3) &= (\mathscr{P}_3 - \mathscr{P}_5) + (\mathscr{P}_3 - \mathscr{P}_6) \\
(\mathscr{P}_1 - \mathscr{P}_4) + (\mathscr{P}_2 - \mathscr{P}_4) &= (\mathscr{P}_4 - \mathscr{P}_B) \\
(\mathscr{P}_1 - \mathscr{P}_5) + (\mathscr{P}_3 - \mathscr{P}_5) &= (\mathscr{P}_5 - \mathscr{P}_B) \\
(\mathscr{P}_2 - \mathscr{P}_6) + (\mathscr{P}_3 - \mathscr{P}_6) &= (\mathscr{P}_6 - \mathscr{P}_B)
\end{aligned}
$$

Solve this system of equations for the modified pressure at each numerical node in terms of the modified pressures at A and B.

$$
\begin{aligned}
\mathscr{P}_1 = \mathscr{P}_2 = \mathscr{P}_3 &= \frac{1}{5}(3\mathscr{P}_A + 2\mathscr{P}_B) \\
\mathscr{P}_4 = \mathscr{P}_5 = \mathscr{P}_6 &= \frac{1}{5}(2\mathscr{P}_A + 3\mathscr{P}_B)
\end{aligned}
$$

Now that these are known, the mass flow rate entering at A can be determined. Use the equation from node A to do this.

$$
\begin{aligned}
w_{\text{A}} &= w_{A1} + w_{A2} + w_{A3} \\
&= \frac{\pi(\mathscr{P}_A - \mathscr{P}_1)R^4\rho}{8\mu L} + \frac{\pi(\mathscr{P}_A - \mathscr{P}_2)R^4\rho}{8\mu L} + \frac{\pi(\mathscr{P}_A - \mathscr{P}_3)R^4\rho}{8\mu L} \\
&= \frac{\pi R^4\rho}{8\mu L}(3\mathscr{P}_A - \mathscr{P}_1 - \mathscr{P}_2 - \mathscr{P}_3) \\
&= \frac{\pi R^4\rho}{8\mu L}\left[3\mathscr{P}_A - \frac{1}{5}(3\mathscr{P}_A + 2\mathscr{P}_B) - \frac{1}{5}(3\mathscr{P}_A + 2\mathscr{P}_B) - \frac{1}{5}(3\mathscr{P}_A + 2\mathscr{P}_B)\right] \\
&= \frac{\pi R^4\rho}{8\mu L}\left[3\mathscr{P}_A - \frac{3}{5}(3\mathscr{P}_A + 2\mathscr{P}_B)\right] \\
&= \frac{\pi R^4\rho}{8\mu L}\left(3\mathscr{P}_A - \frac{9}{5}\mathscr{P}_A - \frac{6}{5}\mathscr{P}_B\right) \\
&= \frac{\pi R^4\rho}{8\mu L}\left(\frac{6}{5}\mathscr{P}_A - \frac{6}{5}\mathscr{P}_B\right) \\
&= \frac{\pi R^4\rho}{8\mu L}\cdot\frac{6}{5}(\mathscr{P}_A - \mathscr{P}_B)
\end{aligned}
$$

Therefore, the mass flow rate of fluid entering at A is

$$
w = \frac{3\pi(\mathscr{P}_A - \mathscr{P}_B)R^4\rho}{20\mu L}.
$$

The same result is obtained for w_B, the rate of mass flowing out of B.