Problem 2B.2

Alternate procedure for solving flow problems. In this chapter we have used the following procedure: (i) derive an equation for the momentum flux, (ii) integrate this equation, (iii) insert Newton's law to get a first-order differential equation for the velocity, (iv) integrate the latter to get the velocity distribution. Another method is: (i) derive an equation for the momentum flux, (ii) insert Newton's law to get a second-order differential equation for the velocity profile, (iii) integrate the latter to get the velocity distribution. Apply this second method to the falling film problem by substituting Eq. 2.2-14 into Eq. 2.2-10 and continuing as directed until the velocity distribution has been obtained and the integration constants evaluated.

Solution

The two equations from the text are as follows.

$$\frac{d\tau_{xz}}{dx} = \rho g \cos\beta \tag{2.2-10}$$

$$\tau_{xz} = -\mu \frac{dv_z}{dx} \tag{2.2-14}$$

Substituting Eq. 2.2-14 into Eq. 2.2-10, we obtain

$$\frac{d}{dx}\left(-\mu \frac{dv_z}{dx}\right) = \rho g \cos\beta$$

$$-\mu \frac{d^2 v_z}{dx^2} = \rho g \cos\beta.$$

This is a second-order differential equation for the velocity profile. Divide both sides by $-\mu$.

$$\frac{d^2 v_z}{dx^2} = -\frac{\rho g \cos\beta}{\mu}$$

There are two boundary conditions—one at the surface $x = 0$ and one at the wall $x = \delta$. Substitute Eq. 2.2-14 into B.C. 1 to get it in terms of v_z.

B.C. 1 (free surface): $\tau_{xz} = 0$ when $x = 0 \quad \rightarrow \quad -\mu \left.\frac{dv_z}{dx}\right|_{x=0} = 0 \quad \rightarrow \quad \left.\frac{dv_z}{dx}\right|_{x=0} = 0$

B.C. 2 (no-slip): $v_z = 0$ when $x = \delta$

Integrate both sides of the differential equation with respect to x.

$$\frac{dv_z}{dx} = -\frac{\rho g \cos\beta}{\mu} x + C_1$$

Apply B.C. 1 here.

$$\left.\frac{dv_z}{dx}\right|_{x=0} = C_1 = 0$$

Integrate both sides of the differential equation with respect to x once more.

$$v_z(x) = -\frac{\rho g \cos\beta}{2\mu} x^2 + C_2$$

Apply B.C. 2 here.

$$v_z(\delta) = -\frac{\rho g \cos\beta}{2\mu}\delta^2 + C_2 = 0 \quad \rightarrow \quad C_2 = \frac{\rho g \cos\beta}{2\mu}\delta^2$$

So we have

$$\begin{aligned} v_z(x) &= -\frac{\rho g \cos\beta}{2\mu}x^2 + \frac{\rho g \cos\beta}{2\mu}\delta^2 \\ &= \frac{\rho g \cos\beta}{2\mu}(\delta^2 - x^2). \end{aligned}$$

Therefore,

$$v_z(x) = \frac{\rho g \delta^2 \cos\beta}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right].$$