Problem 2B.3
Laminar flow in a narrow slit (see Fig. 2B.3).

(a) A Newtonian fluid is in laminar flow in a narrow slit formed by two parallel walls a distance 2B apart. It is understood that B ≪ W, so that “edge effects” are unimportant. Make a differential momentum balance, and obtain the following expressions for the momentum-flux and velocity distributions:

\[ \tau_{xz} = \left( \frac{P_0 - P_L}{L} \right) x \]  \hspace{1cm} (2B.3-1)

\[ v_z = \frac{(P_0 - P_L)B^2}{2\mu L} \left[ 1 - \left( \frac{x}{B} \right)^2 \right] \]  \hspace{1cm} (2B.3-2)

In these expressions \( P = p + \rho gh = p - \rho gz \).

(b) What is the ratio of the average velocity to the maximum velocity for this flow?

(c) Obtain the slit analog of the Hagen–Poiseuille equation.

(d) Draw a meaningful sketch to show why the above analysis is inapplicable if B = W.

(e) How can the result in (b) be obtained from the results of §2.5?

\[ \frac{\langle v_z \rangle}{v_{z,\text{max}}} = \frac{2}{3} \]

\[ w = \frac{2}{3} \frac{(P_0 - P_L)B^3W}{\mu L} \rho \]

Answers: (b) \( \langle v_z \rangle/v_{z,\text{max}} = \frac{2}{3} \)

Solution

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Part (a)

We assume that the fluid flows in the $z$-direction and that its velocity varies as a function of $x$.

$$v_z = v_z(x)$$

As a result, only $\phi_{xz}$ (the $z$-momentum in the positive $x$-direction) and $\phi_{zz}$ (the $z$-momentum in the positive $z$-direction) contribute to the momentum balance. We also assume that the pressure varies with height.

$$p = p(z)$$

Figure 2: This is the shell over which the momentum balance is made for the flow in a slit.

Rate of $z$-momentum into the shell at $z = 0$:  
\[(W \Delta x) \phi_{zz} |_{z=0}\]

Rate of $z$-momentum out of the shell at $z = L$:  
\[(W \Delta x) \phi_{zz} |_{z=L}\]

Rate of $z$-momentum into the shell at $x$:  
\[(W L) \phi_{xz} |_{x}\]

Rate of $z$-momentum out of the shell at $x + \Delta x$:  
\[(W L) \phi_{xz} |_{x+\Delta x}\]

Component of gravitational force on the shell in $z$-direction:  
\[(W L \Delta x) \rho g\]

If we assume steady flow, then the momentum balance is

Rate of momentum in – Rate of momentum out + Force of gravity = 0.

Considering only the $z$-component, we have

\[(W \Delta x) \phi_{zz} |_{z=0} - (W \Delta x) \phi_{zz} |_{z=L} + (W L) \phi_{xz} |_{x} - (W L) \phi_{xz} |_{x+\Delta x} + (W L \Delta x) \rho g = 0.\]

Factor the left side.

\[W \Delta x(\phi_{zz} |_{z=0} - \phi_{zz} |_{z=L}) - WL(\phi_{xz} |_{x+\Delta x} - \phi_{xz} |_{x}) + (W L \Delta x) \rho g = 0\]
Divide both sides by the volume of the shell $WL\Delta x$.

$$\frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} - \frac{\phi_{xz}|_{x+\Delta x} - \phi_{xz}|_{x}}{\Delta x} + \rho g = 0$$

Take the limit as $\Delta x \to 0$.

$$\frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} - \lim_{\Delta x \to 0} \frac{\phi_{xz}|_{x+\Delta x} - \phi_{xz}|_{x}}{\Delta x} + \rho g = 0$$

The second term is the definition of the first derivative.

$$\frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} - \frac{d\phi_{xz}}{dx} + \rho g = 0$$

Now substitute the expressions for $\phi_{xz}$ and $\phi_{zz}$.

$$\phi_{xz} = \tau_{xz} + \rho v_x v_z = \tau_{xz}$$

$$\phi_{zz} = p\delta_{zz} + \tau_{zz} + \rho v_z v_z = p(z) + \rho v_z^2$$

Since $v_z$ does not depend on $z$, the $\rho v_z^2$ terms cancel and we get

$$p(0) + \rho v_z^2|_{z=0} - p(L) - \rho v_z^2|_{z=L} - \frac{d\tau_{xz}}{dx} + \rho g = 0.$$

Make it so $\rho g$ is part of the fraction.

$$\frac{p(0) - p(L) + \rho gL}{L} - \frac{d\tau_{xz}}{dx} = 0$$

Subtract $\rho g0$ from the numerator.

$$\frac{p(0) - \rho g0 - [p(L) - \rho gL]}{L} - \frac{d\tau_{xz}}{dx} = 0$$

Substitute $\mathcal{P}_z = p(z) - \rho gz$.

$$\frac{\mathcal{P}_0 - \mathcal{P}_L}{L} - \frac{d\tau_{xz}}{dx} = 0$$

So we have

$$\frac{d\tau_{xz}}{dx} = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L}.$$

From Newton’s law of viscosity we know that $\tau_{xz} = -\mu(dv_z/dx)$, so

$$\frac{d}{dx} \left( -\mu \frac{dv_z}{dx} \right) = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L}.$$

Bring $-\mu$ in front of the derivative and then divide both sides by it.

$$\frac{d^2v_z}{dx^2} = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu L}$$

We assume the fluid has zero velocity at the walls ($x = \pm B$), i.e. the no-slip boundary condition, and that the maximum velocity occurs furthest from the walls ($x = 0$). That is,

B.C. 1: $\frac{dv_z}{dx} = 0$ when $x = 0$

B.C. 2: $v_z = 0$ when $x = \pm B$. 

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Integrate both sides of the differential equation with respect to \( x \).

\[
\frac{dv_z}{dx} = -\frac{P_0 - P_L}{\mu L}x + C_1
\]

Apply the first boundary condition.

\[
\left. \frac{dv_z}{dx} \right|_{x=0} = C_1 = 0
\]

Integrate both sides of the differential equation with respect to \( x \) once more.

\[
v_z(x) = -\frac{P_0 - P_L}{2\mu L}x^2 + C_2
\]

Apply the second boundary condition.

\[
v_z(\pm B) = -\frac{P_0 - P_L}{2\mu L}B^2 + C_2 = 0 \quad \rightarrow \quad C_2 = \frac{P_0 - P_L}{2\mu L}B^2
\]

With the constants determined, we know the velocity profile.

\[
v_z(x) = -\frac{P_0 - P_L}{2\mu L}x^2 + \frac{P_0 - P_L}{2\mu L}B^2
\]

\[
= \frac{P_0 - P_L}{2\mu L}(B^2 - x^2)
\]

Therefore,

\[
v_z(x) = \frac{P_0 - P_L}{2\mu L}B^2 \left[ 1 - \left( \frac{x}{B} \right)^2 \right]
\]

and

\[
\tau_{xz} = -\mu \frac{dv_z}{dx} = -\frac{P_0 - P_L}{L}x.
\]

**Part (b)**

As mentioned before, the maximum velocity occurs when \( x = 0 \).

\[
v_z(0) = v_{z,\text{max}} = \frac{P_0 - P_L}{2\mu L}B^2
\]

Now we will find the average velocity. It is obtained by integrating \( v_z(x) \) over the area the fluid flows perpendicular to and then dividing by that area.

\[
\langle v_z \rangle = \frac{1}{A} \int v_z \, dA
\]

\[
= \frac{1}{W(2B)} \int_{-B}^{B} \frac{P_0 - P_L}{2\mu L}B^2 \left[ 1 - \left( \frac{x}{B} \right)^2 \right] (W \, dx)
\]

\[
= \frac{B}{4\mu} \frac{P_0 - P_L}{L} \int_{-B}^{B} \left[ 1 - \left( \frac{x}{B} \right)^2 \right] \, dx
\]

\[
= \frac{B}{4\mu} \frac{P_0 - P_L}{L} \left[ x - \frac{x^3}{3B^2} \right]_{-B}^{B}
\]

\[
= \frac{B}{4\mu} \frac{P_0 - P_L}{L} \left( \frac{4}{3} \frac{B}{L} \right)
\]

\[
= \frac{P_0 - P_L}{3\mu L}B^2
\]

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Therefore, the ratio of the average velocity to the maximum velocity is

\[
\frac{\langle v_z \rangle}{v_{z,\text{max}}} = \frac{\frac{P_0 - P_L}{3\mu L} B^2}{\frac{P_0 - P_L}{2\mu L} B^2} = \frac{1}{\frac{3}{2}} = \frac{2}{3}.
\]

**Part (c)**

Here we will find the rate of mass flow \( w \) in the slit. Assume that fluid density \( \rho \) is constant.

\[
w = \frac{dm}{dt} = \frac{d(\rho V)}{dt} = \rho \frac{dV}{dt}
\]

The volumetric flow rate \( dV/dt \) is equal to average velocity times cross-sectional area.

\[
= \rho \langle v_z \rangle (2BW)
\]

\[
= \rho \frac{P_0 - P_L}{3\mu L} B^2 (2BW)
\]

Therefore,

\[
w = \frac{2}{3} \frac{(P_0 - P_L) B^3 W \rho}{\mu L}.
\]

**Part (d)**

Figure 3: This figure illustrates what happens when \( B = W \).

When \( W = B \) hardly any part of the flow is not significantly affected by the edges. The assumption that the velocity only flows in the \( z \)-direction and varies in the \( x \)-direction is no longer reasonable.
Part (e)

§2.5 analyzes flow in a slit lying horizontally for two immiscible fluids with different viscosities ($\mu^I$ and $\mu^H$), so gravity does not appear in the equations there as it does here. The main results from that section for fluid I are the velocity profile,

$$v_I^z = \frac{(p_0 - p_L)b^2}{2\mu^I L} \left[ \left( \frac{2\mu^I}{\mu^I + \mu^H} \right) + \left( \frac{\mu^I - \mu^H}{\mu^I + \mu^H} \right) \left( \frac{x}{b} \right) - \left( \frac{x}{b} \right)^2 \right], \quad (2.5-18)$$

and the average velocity,

$$\langle v_I^z \rangle = \frac{(p_0 - p_L)b^2}{12\mu^I L} \left( \frac{7\mu^I + \mu^H}{\mu^I + \mu^H} \right). \quad (2.5-20)$$

Set $p_0 - p_L = P_0 - P_1$ and set $\mu^I = \mu^H = \mu$ to obtain the corresponding equations for one fluid moving vertically downward in a slit as a result of a pressure difference and gravity. Also, in §2.5 $b$ is used instead of $B$, so set $b = B$.

$$v_I^z = \frac{(P_0 - P_L)B^2}{2\mu L} \left[ 1 - \left( \frac{x}{B} \right)^2 \right]$$

$$\langle v_I^z \rangle = \frac{(P_0 - P_L)B^2}{3\mu L}$$

Set $x = 0$ to obtain the maximum velocity.

$$v_{I,\text{max}} = v_I^z(x = 0) = \frac{(P_0 - P_L)B^2}{2\mu L}$$

The ratio of the average velocity to the maximum velocity is the same as that obtained in part (b).

$$\frac{\langle v_I^z \rangle}{v_{I,\text{max}}} = \frac{(P_0 - P_L)B^2}{3\mu L} \frac{1}{\frac{(P_0 - P_L)B^2}{2\mu L}} = \frac{2}{3} = \frac{4}{6} = \frac{2}{3}$$