Problem 2B.3

Laminar flow in a narrow slit (see Fig. 2B.3).

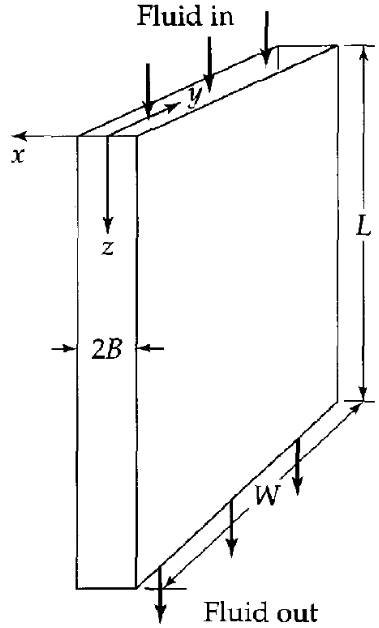

Figure 1: Fig. 2B.3 in the text. Flow through a slit, with $B \ll W \ll L$.

(a) A Newtonian fluid is in laminar flow in a narrow slit formed by two parallel walls a distance $2B$ apart. It is understood that $B \ll W$, so that "edge effects" are unimportant. Make a differential momentum balance, and obtain the following expressions for the momentum-flux and velocity distributions:

$$\tau_{xz} = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{L}\right) x \tag{2B.3-1}$$

$$v_z = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right] \tag{2B.3-2}$$

In these expressions $\mathscr{P} = p + \rho g h = p - \rho g z$.

(b) What is the ratio of the average velocity to the maximum velocity for this flow?

(c) Obtain the slit analog of the Hagen–Poiseuille equation.

(d) Draw a meaningful sketch to show why the above analysis is inapplicable if $B = W$.

(e) How can the result in (b) be obtained from the results of §2.5?

Answers: **(b)** $\langle v_z \rangle / v_{z,\max} = \dfrac{2}{3}$

(c) $w = \dfrac{2}{3}\dfrac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W \rho}{\mu L}$

Solution

Part (a)

We assume that the fluid flows in the z-direction and that its velocity varies as a function of x.

$$v_z = v_z(x)$$

As a result, only ϕ_{xz} (the z-momentum in the positive x-direction) and ϕ_{zz} (the z-momentum in the positive z-direction) contribute to the momentum balance. We also assume that the pressure varies with height.

$$p = p(z)$$

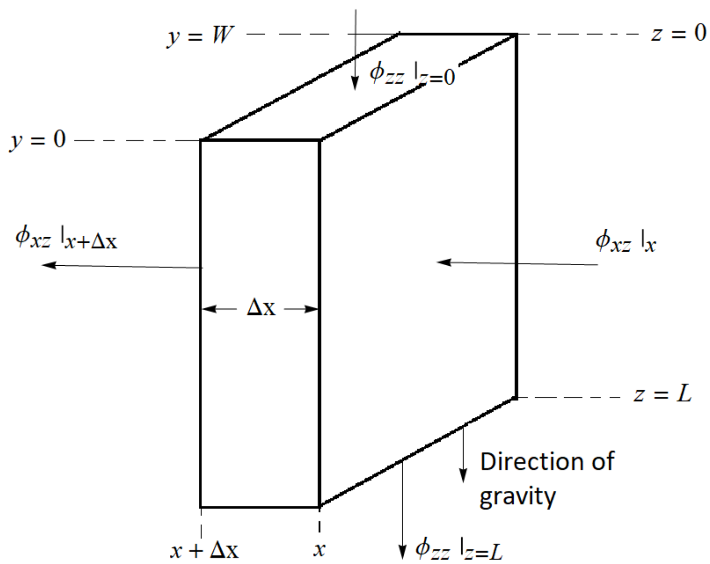

Figure 2: This is the shell over which the momentum balance is made for the flow in a slit.

Rate of z-momentum into the shell at $z = 0$:	$(W\Delta x)\phi_{zz}\|_{z=0}$
Rate of z-momentum out of the shell at $z = L$:	$(W\Delta x)\phi_{zz}\|_{z=L}$
Rate of z-momentum into the shell at x:	$(WL)\phi_{xz}\|_{x}$
Rate of z-momentum out of the shell at $x + \Delta x$:	$(WL)\phi_{xz}\|_{x+\Delta x}$
Component of gravitational force on the shell in z-direction:	$(WL\Delta x)\rho g$

If we assume steady flow, then the momentum balance is

$$\text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = \mathbf{0}.$$

Considering only the z-component, we have

$$(W\Delta x)\phi_{zz}|_{z=0} - (W\Delta x)\phi_{zz}|_{z=L} + (WL)\phi_{xz}|_{x} - (WL)\phi_{xz}|_{x+\Delta x} + (WL\Delta x)\rho g = 0.$$

Factor the left side.

$$W\Delta x(\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}) - WL(\phi_{xz}|_{x+\Delta x} - \phi_{xz}|_{x}) + (WL\Delta x)\rho g = 0$$

Divide both sides by the volume of the shell $WL\Delta x$.

$$\frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} - \frac{\phi_{xz}|_{x+\Delta x} - \phi_{xz}|_x}{\Delta x} + \rho g = 0$$

Take the limit as $\Delta x \to 0$.

$$\frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} - \lim_{\Delta x \to 0} \frac{\phi_{xz}|_{x+\Delta x} - \phi_{xz}|_x}{\Delta x} + \rho g = 0$$

The second term is the definition of the first derivative.

$$\frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} - \frac{d\phi_{xz}}{dx} + \rho g = 0$$

Now substitute the expressions for ϕ_{xz} and ϕ_{zz}.

$$\begin{aligned}\phi_{xz} &= \tau_{xz} + \cancel{\rho v_x v_z} = \tau_{xz}\\ \phi_{zz} &= p\delta_{zz} + \cancel{\tau_{zz}} + \rho v_z v_z = p(z) + \rho v_z^2\end{aligned}$$

Since v_z does not depend on z, the ρv_z^2 terms cancel and we get

$$\frac{p(0) + \cancel{\rho v_z^2|_{z=0}} - p(L) - \cancel{\rho v_z^2|_{z=L}}}{L} - \frac{d\tau_{xz}}{dx} + \rho g = 0.$$

Make it so ρg is part of the fraction.

$$\frac{p(0) - p(L) + \rho g L}{L} - \frac{d\tau_{xz}}{dx} = 0$$

Subtract $\rho g 0$ from the numerator.

$$\frac{p(0) - \rho g 0 - [p(L) - \rho g L]}{L} - \frac{d\tau_{xz}}{dx} = 0$$

Substitute $\mathscr{P}_z = p(z) - \rho g z$.

$$\frac{\mathscr{P}_0 - \mathscr{P}_L}{L} - \frac{d\tau_{xz}}{dx} = 0$$

So we have

$$\frac{d\tau_{xz}}{dx} = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L}.$$

From Newton's law of viscosity we know that $\tau_{xz} = -\mu(dv_z/dx)$, so

$$\frac{d}{dx}\left(-\mu \frac{dv_z}{dx}\right) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L}.$$

Bring $-\mu$ in front of the derivative and then divide both sides by it.

$$\frac{d^2 v_z}{dx^2} = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{\mu L}$$

We assume the fluid has zero velocity at the walls ($x = \pm B$), i.e. the no-slip boundary condition, and that the maximum velocity occurs furthest from the walls ($x = 0$). That is,

$$\begin{aligned}&\text{B.C. 1:} \quad \frac{dv_z}{dx} = 0 \quad \text{when} \quad x = 0\\ &\text{B.C. 2:} \quad v_z = 0 \quad \text{when} \quad x = \pm B.\end{aligned}$$

Integrate both sides of the differential equation with respect to x.

$$\frac{dv_z}{dx} = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{\mu L}x + C_1$$

Apply the first boundary condition.

$$\left.\frac{dv_z}{dx}\right|_{x=0} = C_1 = 0$$

Integrate both sides of the differential equation with respect to x once more.

$$v_z(x) = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}x^2 + C_2$$

Apply the second boundary condition.

$$v_z(\pm B) = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}B^2 + C_2 = 0 \quad \rightarrow \quad C_2 = \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}B^2$$

With the constants determined, we know the velocity profile.

$$\begin{aligned} v_z(x) &= -\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}x^2 + \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}B^2 \\ &= \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}(B^2 - x^2) \end{aligned}$$

Therefore,

$$v_z(x) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}B^2\left[1 - \left(\frac{x}{B}\right)^2\right]$$

and

$$\tau_{xz} = -\mu\frac{dv_z}{dx} = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{L}x.$$

Part (b)

As mentioned before, the maximum velocity occurs when $x = 0$.

$$v_z(0) = v_{z,\max} = \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}B^2$$

Now we will find the average velocity. It is obtained by integrating $v_z(x)$ over the area the fluid flows perpendicular to and then dividing by that area.

$$\begin{aligned} \langle v_z \rangle &= \frac{1}{A}\int v_z\, dA \\ &= \frac{1}{W(2B)}\int_{-B}^{B}\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}B^2\left[1 - \left(\frac{x}{B}\right)^2\right](W\, dx) \\ &= \frac{B}{4\mu}\frac{\mathscr{P}_0 - \mathscr{P}_L}{L}\int_{-B}^{B}\left[1 - \left(\frac{x}{B}\right)^2\right]dx \\ &= \frac{B}{4\mu}\frac{\mathscr{P}_0 - \mathscr{P}_L}{L}\left.\left(x - \frac{x^3}{3B^2}\right)\right|_{-B}^{B} \\ &= \frac{B}{4\mu}\frac{\mathscr{P}_0 - \mathscr{P}_L}{L}\left(\frac{4}{3}B\right) \\ &= \frac{\mathscr{P}_0 - \mathscr{P}_L}{3\mu L}B^2 \end{aligned}$$

Therefore, the ratio of the average velocity to the maximum velocity is

$$\frac{\langle v_z \rangle}{v_{z,\max}} = \frac{\frac{\mathscr{P}_0 - \mathscr{P}_L}{3\mu L} B^2}{\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L} B^2} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

Part (c)

Here we will find the rate of mass flow w in the slit. Assume that fluid density ρ is constant.

$$w = \frac{dm}{dt} = \frac{d(\rho V)}{dt} = \rho \frac{dV}{dt}$$

The volumetric flow rate dV/dt is equal to average velocity times cross-sectional area.

$$\begin{aligned} &= \rho \langle v_z \rangle (2BW) \\ &= \rho \frac{\mathscr{P}_0 - \mathscr{P}_L}{3\mu L} B^2 (2BW) \end{aligned}$$

Therefore,

$$w = \frac{2}{3} \frac{(\mathscr{P}_0 - \mathscr{P}_L) B^3 W \rho}{\mu L}.$$

Part (d)

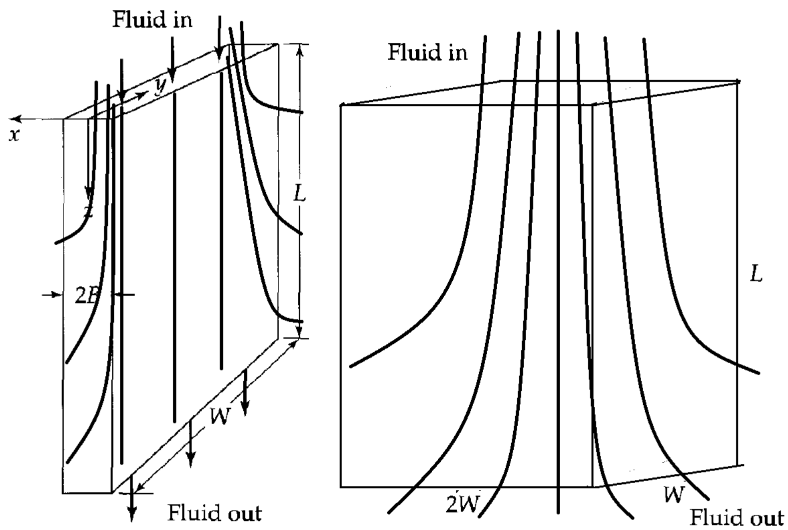

Figure 3: This figure illustrates what happens when $B = W$.

When $W = B$ hardly any part of the flow is not signficantly affected by the edges. The assumption that the velocity only flows in the z-direction and varies in the x-direction is no longer reasonable.

Part (e)

§2.5 analyzes flow in a slit lying horizontally for two immiscible fluids with different viscosities (μ^{I} and μ^{II}), so gravity does not appear in the equations there as it does here. The main results from that section for fluid I are the velocity profile,

$$v_z^{\text{I}} = \frac{(p_0 - p_L)b^2}{2\mu^{\text{I}} L}\left[\left(\frac{2\mu^{\text{I}}}{\mu^{\text{I}} + \mu^{\text{II}}}\right) + \left(\frac{\mu^{\text{I}} - \mu^{\text{II}}}{\mu^{\text{I}} + \mu^{\text{II}}}\right)\left(\frac{x}{b}\right) - \left(\frac{x}{b}\right)^2\right], \tag{2.5-18}$$

and the average velocity,

$$\langle v_z^{\text{I}} \rangle = \frac{(p_0 - p_L)b^2}{12\mu^{\text{I}} L}\left(\frac{7\mu^{\text{I}} + \mu^{\text{II}}}{\mu^{\text{I}} + \mu^{\text{II}}}\right). \tag{2.5-20}$$

Set $p_0 - p_L = \mathscr{P}_0 - \mathscr{P}_1$ and set $\mu^{\text{I}} = \mu^{\text{II}} = \mu$ to obtain the corresponding equations for one fluid moving vertically downward in a slit as a result of a pressure difference and gravity. Also, in §2.5 b is used instead of B, so set $b = B$.

$$\begin{aligned} v_z^{\text{I}} &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right] \\ \langle v_z^{\text{I}} \rangle &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{3\mu L} \end{aligned}$$

Set $x = 0$ to obtain the maximum velocity.

$$v_{z,\max}^{\text{I}} = v_z^{\text{I}}(x = 0) = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L}$$

The ratio of the average velocity to the maximum velocity is the same as that obtained in part (b).

$$\frac{\langle v_z^{\text{I}} \rangle}{v_{z,\max}^{\text{I}}} = \frac{\frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{3\mu L}}{\frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$