Problem 2B.4

Laminar slit flow with a moving wall ("plane Couette flow"). Extend Problem 2B.3 by allowing the wall at $x = B$ to move in the positive z direction at a steady speed v_0. Obtain (a) the shear-stress distribution and (b) the velocity distribution. Draw carefully labeled sketches of these functions.

$$\textit{Answers: } \tau_{xz} = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{L}\right) x - \frac{\mu v_0}{2B};\ v_z = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right] + \frac{v_0}{2}\left(1 + \frac{x}{B}\right)$$

Solution

The analysis in this problem is the same as in Problem 2B.3 except that the boundary conditions are different. Here the velocity at $x = B$ is equal to v_0, and the velocity at $x = -B$ is equal to 0.

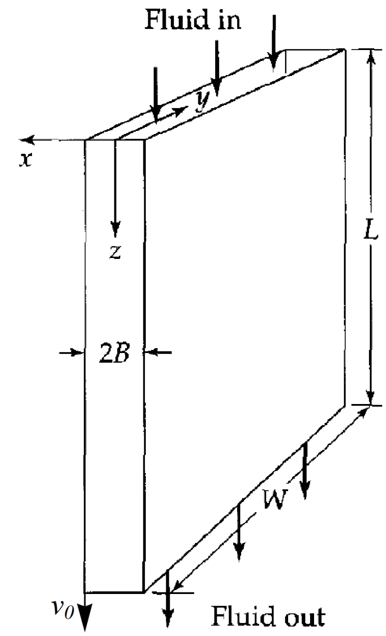

Figure 1: This is Fig. 2B.3 with the wall at $x = B$ moving in the positive z-direction with speed v_0. Fluid is flowing through the slit with $B \ll W \ll L$.

We assume that the fluid flows in the z-direction and that its velocity varies as a function of x.

$$v_z = v_z(x)$$

As a result, only ϕ_{xz} (the z-momentum in the positive x-direction) and ϕ_{zz} (the z-momentum in the positive z-direction) contribute to the momentum balance. Also, the boundary conditions are as follows.

$$\begin{aligned} &\text{B.C. 1:} \quad v_z = 0 \quad \text{when} \quad x = -B \\ &\text{B.C. 2:} \quad v_z = v_0 \quad \text{when} \quad x = B. \end{aligned}$$

We also assume the pressure varies with height.

$$p = p(z)$$

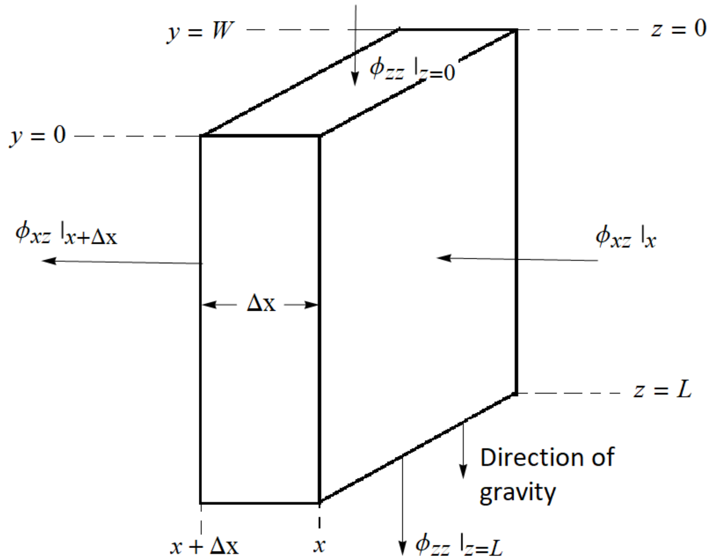

Figure 2: This is the shell over which the momentum balance is made for the flow in a slit.

Rate of z-momentum into the shell at $z = 0$:	$(W\Delta x)\phi_{zz}\|_{z=0}$
Rate of z-momentum out of the shell at $z = L$:	$(W\Delta x)\phi_{zz}\|_{z=L}$
Rate of z-momentum into the shell at x:	$(WL)\phi_{xz}\|_{x}$
Rate of z-momentum out of the shell at $x + \Delta x$:	$(WL)\phi_{xz}\|_{x+\Delta x}$
Component of gravitational force on the shell in z-direction:	$(WL\Delta x)\rho g$

If we assume steady flow, then the momentum balance is

$$\text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = \mathbf{0}.$$

Considering only the z-component, we have

$$(W\Delta x)\phi_{zz}|_{z=0} - (W\Delta x)\phi_{zz}|_{z=L} + (WL)\phi_{xz}|_{x} - (WL)\phi_{xz}|_{x+\Delta x} + (WL\Delta x)\rho g = 0.$$

Factor the left side.

$$W\Delta x(\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}) - WL(\phi_{xz}|_{x+\Delta x} - \phi_{xz}|_{x}) + (WL\Delta x)\rho g = 0$$

Divide both sides by the volume of the shell $WL\Delta x$.

$$\frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} - \frac{\phi_{xz}|_{x+\Delta x} - \phi_{xz}|_{x}}{\Delta x} + \rho g = 0$$

Take the limit as $\Delta x \to 0$.

$$\frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} - \lim_{\Delta x \to 0} \frac{\phi_{xz}|_{x+\Delta x} - \phi_{xz}|_{x}}{\Delta x} + \rho g = 0$$

The second term is the definition of the first derivative.

$$\frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} - \frac{d\phi_{xz}}{dx} + \rho g = 0$$

Now substitute the expressions for ϕ_{xz} and ϕ_{zz}.

$$\phi_{xz} = \tau_{xz} + \cancel{\rho v_x v_z} = \tau_{xz}$$
$$\phi_{zz} = p\delta_{zz} + \cancel{\tau_{zz}} + \rho v_z v_z = p(z) + \rho v_z^2$$

Since v_z does not depend on z, the ρv_z^2 terms cancel and we get

$$\frac{p(0) + \cancel{\rho v_z^2\big|_{z=0}} - p(L) - \cancel{\rho v_z^2\big|_{z=L}}}{L} - \frac{d\tau_{xz}}{dx} + \rho g = 0.$$

Make it so ρg is part of the fraction.

$$\frac{p(0) - p(L) + \rho g L}{L} - \frac{d\tau_{xz}}{dx} = 0$$

Subtract $\rho g 0$ from the numerator.

$$\frac{p(0) - \rho g 0 - [p(L) - \rho g L]}{L} - \frac{d\tau_{xz}}{dx} = 0$$

Substitute $\mathscr{P}_z = p(z) - \rho g z$.

$$\frac{\mathscr{P}_0 - \mathscr{P}_L}{L} - \frac{d\tau_{xz}}{dx} = 0$$

So we have

$$\frac{d\tau_{xz}}{dx} = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L}.$$

From Newton's law of viscosity we know that $\tau_{xz} = -\mu(dv_z/dx)$, so

$$\frac{d}{dx}\left(-\mu \frac{dv_z}{dx}\right) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L}.$$

Bring $-\mu$ in front of the derivative and then divide both sides by it.

$$\frac{d^2 v_z}{dx^2} = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{\mu L}$$

Integrate both sides of the differential equation with respect to x.

$$\frac{dv_z}{dx} = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{\mu L} x + C_1$$

Integrate both sides of the differential equation with respect to x once more.

$$v_z(x) = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L} x^2 + C_1 x + C_2$$

Apply the boundary conditions now to determine C_1 and C_2.

$$v_z(-B) = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L} B^2 - C_1 B + C_2 = 0$$
$$v_z(B) = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L} B^2 + C_1 B + C_2 = v_0$$

Solving this system of equations for C_1 and C_2 gives

$$C_1 = \frac{v_0}{2B}$$
$$C_2 = \frac{v_0}{2} + \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L} B^2,$$

so we get

$$v_z(x) = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L} x^2 + \frac{v_0}{2B} x + \frac{v_0}{2} + \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L} B^2.$$

Factor this result.

$$v_z(x) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}(B^2 - x^2) + \frac{v_0}{2}\left(1 + \frac{x}{B}\right)$$
$$= \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L} B^2 \left(1 - \frac{x^2}{B^2}\right) + \frac{v_0}{2}\left(1 + \frac{x}{B}\right)$$

Therefore,

$$\boxed{v_z = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right] + \frac{v_0}{2}\left(1 + \frac{x}{B}\right).}$$

Since $\tau_{xz} = -\mu(dv_z/dx)$, we have

$$\tau_{xz} = -\mu\left(-\frac{\mathscr{P}_0 - \mathscr{P}_L}{\mu L} x + \frac{v_0}{2B}\right).$$

Therefore,

$$\boxed{\tau_{xz} = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L} x - \frac{\mu v_0}{2B}.}$$

The maximum velocity can be obtained by taking the derivative of $v_z(x)$ and setting it equal to zero.

$$\frac{dv_z}{dx} = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{\mu L} x + \frac{v_0}{2B} = 0$$

Solving for x yields the x-coordinate where the maximum occurs.

$$x = \frac{\mu L v_0}{2B(\mathscr{P}_0 - \mathscr{P}_L)}$$

Plugging this into $v_z(x)$ and simplifying, we get

$$v_z\left(\frac{\mu L v_0}{2B(\mathscr{P}_0 - \mathscr{P}_L)}\right) = \frac{[2B^2(\mathscr{P}_0 - \mathscr{P}_L) + \mu L v_0]^2}{8B^2 \mu L(\mathscr{P}_0 - \mathscr{P}_L)}$$

for the maximum velocity.

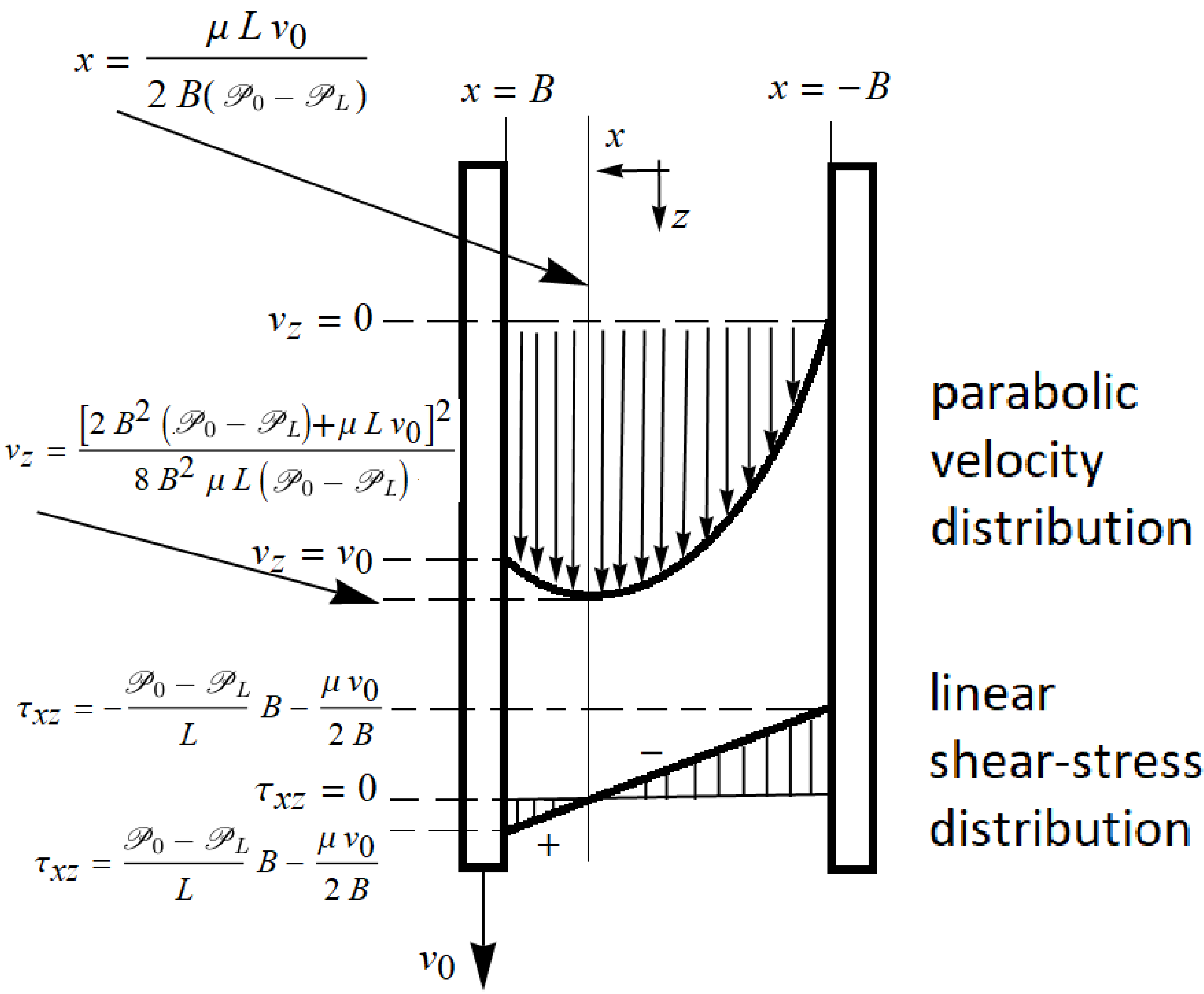

Figure 3: This is a carefully labeled sketch of the velocity distribution $v_z(x)$ and the shear-stress (momentum-flux) distribution $\tau_{xz}(x)$ for flow in a slit with one wall moving at speed v_0.