Problem 2B.4

Laminar slit flow with a moving wall (“plane Couette flow”). Extend Problem 2B.3 by allowing the wall at \( x = B \) to move in the positive \( z \) direction at a steady speed \( v_0 \). Obtain \( a \) the shear-stress distribution and \( b \) the velocity distribution. Draw carefully labeled sketches of these functions.

\[
\text{Answers: } \tau_{xz} = \left( \frac{\rho_0 - \rho_L}{L} \right) x - \frac{\mu v_0}{2B}; \quad v_z = \left( \frac{\rho_0 - \rho_L}{2\mu L} \right) B^2 \left[ 1 - \left( \frac{x}{B} \right)^2 \right] + \frac{v_0}{2} \left( 1 + \frac{x}{B} \right)
\]

Solution

The analysis in this problem is the same as in Problem 2B.3 except that the boundary conditions are different. Here the velocity at \( x = B \) is equal to \( v_0 \), and the velocity at \( x = -B \) is equal to 0.

We assume that the fluid flows in the \( z \)-direction and that its velocity varies as a function of \( x \).

\[
v_z = v_z(x)
\]

As a result, only \( \phi_{xz} \) (the \( z \)-momentum in the positive \( x \)-direction) and \( \phi_{zz} \) (the \( z \)-momentum in the positive \( z \)-direction) contribute to the momentum balance. Also, the boundary conditions are as follows.

\[
\text{B.C. 1: } \quad v_z = 0 \quad \text{when} \quad x = -B \\
\text{B.C. 2: } \quad v_z = v_0 \quad \text{when} \quad x = B.
\]

We also assume the pressure varies with height.

\[
p = p(z)
\]
Figure 2: This is the shell over which the momentum balance is made for the flow in a slit.

Rate of \( z \)-momentum into the shell at \( z = 0 \): \( (W \Delta x)\phi_{zz} \mid_{z=0} \)

Rate of \( z \)-momentum out of the shell at \( z = L \): \( (W \Delta x)\phi_{zz} \mid_{z=L} \)

Rate of \( z \)-momentum into the shell at \( x \): \( (WL)\phi_{xz} \mid_{x} \)

Rate of \( z \)-momentum out of the shell at \( x + \Delta x \): \( (WL)\phi_{xz} \mid_{x+\Delta x} \)

Component of gravitational force on the shell in \( z \)-direction: \( (WL\Delta x)\rho g \)

If we assume steady flow, then the momentum balance is

\[
\text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = 0.
\]

Considering only the \( z \)-component, we have

\[
(W \Delta x)(\phi_{zz} \mid_{z=0} - (W \Delta x)\phi_{zz} \mid_{z=L}) + (WL)\phi_{xz} \mid_{x} - (WL)\phi_{xz} \mid_{x+\Delta x} + (WL\Delta x)\rho g = 0.
\]

Factor the left side.

\[
W \Delta x(\phi_{zz} \mid_{z=0} - \phi_{zz} \mid_{z=L}) - WL(\phi_{xz} \mid_{x+\Delta x} - \phi_{xz} \mid_{x}) + (WL\Delta x)\rho g = 0.
\]

Divide both sides by the volume of the shell \( WL\Delta x \).

\[
\frac{\phi_{zz} \mid_{z=0} - \phi_{zz} \mid_{z=L}}{L} - \frac{\phi_{xz} \mid_{x+\Delta x} - \phi_{xz} \mid_{x}}{\Delta x} + \rho g = 0
\]

Take the limit as \( \Delta x \to 0 \).

\[
\frac{\phi_{zz} \mid_{z=0} - \phi_{zz} \mid_{z=L}}{L} - \lim_{\Delta x \to 0} \frac{\phi_{xz} \mid_{x+\Delta x} - \phi_{xz} \mid_{x}}{\Delta x} + \rho g = 0
\]

The second term is the definition of the first derivative.

\[
\frac{\phi_{zz} \mid_{z=0} - \phi_{zz} \mid_{z=L}}{L} - \frac{d\phi_{xz}}{dx} + \rho g = 0
\]

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Now substitute the expressions for $\phi_{xz}$ and $\phi_{zz}$.

$\phi_{xz} = \tau_{xz} + \rho v_z^2 v_z = \tau_{xz}$

$\phi_{zz} = p\delta_{zz} + \tau_{zz} + \rho v_z v_z = p(z) + \rho v_z^2$

Since $v_z$ does not depend on $z$, the $\rho v_z^2$ terms cancel and we get

$$p(0) + \frac{\rho v_z^2}{z=0} - p(L) - \frac{\rho v_z^2}{z=L} = \frac{d\tau_{xz}}{dx} + \rho g = 0.$$  

Make it so $\rho g$ is part of the fraction.

$$\frac{p(0) - p(L) + \rho g L}{L} = \frac{d\tau_{xz}}{dx} = 0$$

Subtract $\rho g0$ from the numerator.

$$\frac{p(0) - \rho g0 - [p(L) - \rho gL]}{L} = \frac{d\tau_{xz}}{dx} = 0$$

Substitute $\mathcal{P}_z = p(z) - \rho g z$.

$$\frac{\mathcal{P}_0 - \mathcal{P}_L}{L} = \frac{d\tau_{xz}}{dx} = 0$$

So we have

$$\frac{d\tau_{xz}}{dx} = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L}.$$  

From Newton’s law of viscosity we know that $\tau_{xz} = -\mu (dv_z/dx)$, so

$$\frac{d}{dx} \left( -\mu \frac{dv_z}{dx} \right) = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L}.$$  

Bring $-\mu$ in front of the derivative and then divide both sides by it.

$$\frac{d^2 v_z}{dx^2} = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu L}$$

Integrate both sides of the differential equation with respect to $x$.

$$\frac{dv_z}{dx} = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu L} x + C_1$$

Integrate both sides of the differential equation with respect to $x$ once more.

$$v_z(x) = \frac{\mathcal{P}_0 - \mathcal{P}_L}{2\mu L} x^2 + C_1 x + C_2$$

Apply the boundary conditions now to determine $C_1$ and $C_2$.

$$v_z(-B) = \frac{\mathcal{P}_0 - \mathcal{P}_L}{2\mu L} B^2 - C_1 B + C_2 = 0$$

$$v_z(B) = \frac{\mathcal{P}_0 - \mathcal{P}_L}{2\mu L} B^2 + C_1 B + C_2 = v_0$$

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Solving this system of equations for $C_1$ and $C_2$ gives

$$C_1 = \frac{v_0}{2B}$$

$$C_2 = \frac{v_0}{2} + \frac{P_0 - P_L}{2\mu L} B^2,$$

so we get

$$v_z(x) = -\frac{P_0 - P_L}{2\mu L} x^2 + \frac{v_0}{2B} x + \frac{v_0}{2} + \frac{P_0 - P_L}{2\mu L} B^2.$$

Factor this result.

$$v_z(x) = \frac{P_0 - P_L}{2\mu L} (B^2 - x^2) + \frac{v_0}{2} \left( 1 + \frac{x}{B} \right)$$

$$= \frac{P_0 - P_L}{2\mu L} B^2 \left( 1 - \frac{x^2}{B^2} \right) + \frac{v_0}{2} \left( 1 + \frac{x}{B} \right)$$

Therefore,

$$v_z = \frac{(P_0 - P_L)B^2}{2\mu L} \left[ 1 - \left( \frac{x}{B} \right)^2 \right] + \frac{v_0}{2} \left( 1 + \frac{x}{B} \right).$$

Since $\tau_{xz} = -\mu (dv_z/dx)$, we have

$$\tau_{xz} = -\mu \left( -\frac{P_0 - P_L}{\mu L} x + \frac{v_0}{2B} \right).$$

Therefore,

$$\tau_{xz} = \frac{P_0 - P_L}{L} x - \frac{\mu v_0}{2B}.$$

The maximum velocity can be obtained by taking the derivative of $v_z(x)$ and setting it equal to zero.

$$\frac{dv_z}{dx} = -\frac{P_0 - P_L}{\mu L} x + \frac{v_0}{2B} = 0$$

Solving for $x$ yields the $x$-coordinate where the maximum occurs.

$$x = \frac{\mu L v_0}{2B(P_0 - P_L)}$$

Plugging this into $v_z(x)$ and simplifying, we get

$$v_z \left( \frac{\mu L v_0}{2B(P_0 - P_L)} \right) = \frac{[2B^2(P_0 - P_L) + \mu L v_0]^2}{8B^2 \mu L (P_0 - P_L)}$$

for the maximum velocity.
Figure 3: This is a carefully labeled sketch of the velocity distribution \( v_z(x) \) and the shear-stress (momentum-flux) distribution \( \tau_{xz}(x) \) for flow in a slit with one wall moving at speed \( v_0 \).