Problem 2B.5

Interrelation of slit and annulus formulas. When an annulus is very thin, it may, to a good approximation, be considered as a thin slit. Then the results of Problem 2B.3 can be taken over with suitable modifications. For example, the mass rate of flow in an annulus with outer wall of radius R and inner wall of radius $(1-\varepsilon)R$, where ε is small, may be obtained from Problem 2B.3 by replacing $2B$ by εR, and W by $2\pi(1-\frac{1}{2}\varepsilon)R$. In this way we get for the mass rate of flow:

$$w = \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4\varepsilon^3\rho}{6\mu L}\left(1 - \frac{1}{2}\varepsilon\right) \tag{2B.5-1}$$

Show that this same result may be obtained from Eq. 2.4-17 by setting κ equal to $1-\varepsilon$ everywhere in the formula and then expanding the expression for w in powers of ε. This requires using the Taylor series (see §C.2)

$$\ln(1-\varepsilon) = -\varepsilon - \frac{1}{2}\varepsilon^2 - \frac{1}{3}\varepsilon^3 - \frac{1}{4}\varepsilon^4 - \cdots \tag{2B.5-2}$$

and then performing a long division. The first term in the resulting series will be Eq. 2B.5-1. *Caution:* In the derivation it is necessary to use the first *four* terms of the Taylor series in Eq. 2B.5-2.

Solution

Eq. 2.4-17 in the text,

$$w = \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4\rho}{8\mu L}\left[(1-\kappa^4) - \frac{(1-\kappa^2)^2}{\ln(1/\kappa)}\right], \tag{2.4-17}$$

gives the mass flow rate w in an annulus. Set $\kappa = 1-\varepsilon$ as instructed.

$$w = \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4\rho}{8\mu L}\left\{[1-(1-\varepsilon)^4] - \frac{[1-(1-\varepsilon)^2]^2}{\ln[1/(1-\varepsilon)]}\right\}$$

Expand the first term in curly braces and the numerator of the fraction. Also, use the minus sign in front of the fraction to invert the logarithm's argument.

$$w = \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4\rho}{8\mu L}\left[4\varepsilon - 6\varepsilon^2 + 4\varepsilon^3 - \varepsilon^4 + \frac{4\varepsilon^2 - 4\varepsilon^3 + \varepsilon^4}{\ln(1-\varepsilon)}\right]$$

Substitute the Taylor series expansion for $\ln(1-\varepsilon)$, making sure to use the first four terms.

$$w = \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4\rho}{8\mu L}\left(4\varepsilon - 6\varepsilon^2 + 4\varepsilon^3 - \varepsilon^4 + \frac{4\varepsilon^2 - 4\varepsilon^3 + \varepsilon^4}{-\varepsilon - \frac{1}{2}\varepsilon^2 - \frac{1}{3}\varepsilon^3 - \frac{1}{4}\varepsilon^4 - \cdots}\right)$$

Factor out the minus sign in the denominator.

$$w = \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4\rho}{8\mu L}\left(4\varepsilon - 6\varepsilon^2 + 4\varepsilon^3 - \varepsilon^4 - \frac{4\varepsilon^2 - 4\varepsilon^3 + \varepsilon^4}{\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \frac{1}{4}\varepsilon^4 + \cdots}\right)$$

Proceed with the long division.

$$\begin{array}{rl}
 & 4\varepsilon - 6\varepsilon^2 + \frac{8}{3}\varepsilon^3 - \frac{1}{3}\varepsilon^4 + \cdots \\
\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \frac{1}{4}\varepsilon^4 + \cdots \Big) & \overline{4\varepsilon^2 - 4\varepsilon^3 + \varepsilon^4 + 0\varepsilon^5} \\
(-) & 4\varepsilon^2 + 2\varepsilon^3 + \frac{4}{3}\varepsilon^4 + \varepsilon^5 + \cdots \\
 & \overline{-6\varepsilon^3 - \frac{1}{3}\varepsilon^4 - \varepsilon^5 - \cdots} \\
(-) & -6\varepsilon^3 - 3\varepsilon^4 - 2\varepsilon^5 - \cdots \\
 & \overline{\frac{8}{3}\varepsilon^4 + \varepsilon^5 + \cdots} \\
(-) & \frac{8}{3}\varepsilon^4 + \frac{4}{3}\varepsilon^5 + \cdots \\
 & \overline{-\frac{1}{3}\varepsilon^5 - \cdots}
\end{array}$$

The first four terms of the quotient will suffice.

$$\begin{aligned}
w &= \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4\rho}{8\mu L}\left[4\varepsilon - 6\varepsilon^2 + 4\varepsilon^3 - \varepsilon^4 - \left(4\varepsilon - 6\varepsilon^2 + \frac{8}{3}\varepsilon^3 - \frac{1}{3}\varepsilon^4 + \cdots\right)\right] \\
&= \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4\rho}{8\mu L}\left(\cancel{4\varepsilon} - \cancel{6\varepsilon^2} + 4\varepsilon^3 - \varepsilon^4 - \cancel{4\varepsilon} + \cancel{6\varepsilon^2} - \frac{8}{3}\varepsilon^3 + \frac{1}{3}\varepsilon^4 - \cdots\right) \\
&= \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4\rho}{8\mu L}\left(\frac{4}{3}\varepsilon^3 - \frac{2}{3}\varepsilon^4 - \cdots\right) \\
&= \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4\rho}{8\mu L}\frac{4}{3}\varepsilon^3\left(1 - \frac{1}{2}\varepsilon - \cdots\right) \\
&= \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4\varepsilon^3\rho}{6\mu L}\left(1 - \frac{1}{2}\varepsilon\right) - \cdots
\end{aligned}$$

Keeping only the first term in the series, we therefore have Eq. 2B.5-1.

$$w = \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4\varepsilon^3\rho}{6\mu L}\left(1 - \frac{1}{2}\varepsilon\right)$$