Problem 2B.8

Analysis of a capillary flowmeter (see Fig. 2B.8).

Determine the rate of flow (in lb m/hr) through the capillary flow meter shown in the figure. The fluid flowing in the inclined tube is water at 20°C, and the manometer fluid is carbon tetrachloride (CCl₄) with density 1.594 g/cm³. The capillary diameter is 0.010 in. Note: Measurements of \( H \) and \( L \) are sufficient to calculate the flow rate; \( \theta \) need not be measured. Why?

Solution

For the capillary flow meter, choose a cylindrical coordinate system with the positive \( z \)-direction pointing in the direction of the flow. The fluid velocity is assumed to vary as a function of radius \( r \).

\[ v_z = v_z(r) \]

As a result, only \( \phi_{rz} \) (the \( z \)-momentum in the positive \( r \)-direction) and \( \phi_{zz} \) (the \( z \)-momentum in the positive \( z \)-direction) contribute to the momentum balance. Figure 1 on the next page shows the shell the momentum balance is made over.

Rate of \( z \)-momentum into the shell at \( z = 0 \): \[ (2\pi r \Delta r) \phi_{zz}|_{z=0} \]
Rate of \( z \)-momentum out of the shell at \( z = L \): \[ (2\pi r \Delta r) \phi_{zz}|_{z=L} \]
Rate of \( z \)-momentum into the shell at \( r \): \[ (2\pi r L) \phi_{rz}|_r \]
Rate of \( z \)-momentum out of the shell at \( r + \Delta r \): \[ [2\pi (r + \Delta r) L] \phi_{rz}|_{r+\Delta r} \]
Component of gravitational force on the shell in \( z \)-direction: \[ (2\pi r \Delta r L) \rho g \sin \theta \]

If we assume steady flow, then the momentum balance is

\[ \text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = 0 \]

Considering only the \( z \)-component, we have

\[ (2\pi r \Delta r) \phi_{zz}|_{z=0} - (2\pi r \Delta r) \phi_{zz}|_{z=L} + (2\pi r L) \phi_{rz}|_r - [2\pi (r + \Delta r) L] \phi_{rz}|_{r+\Delta r} + (2\pi r \Delta r L) \rho g \sin \theta = 0. \]
Figure 1: This is the shell over which the momentum balance is made for fluid going through the capillary flow meter.

Factor the left side.

\[-2\pi r \Delta r (\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}) - 2\pi L [(r + \Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r] + 2\pi r \Delta r L \rho g \sin \theta = 0\]

Divide both sides by \(2\pi \Delta r L\).

\[-r \phi_{zz}|_{z=L} - \phi_{zz}|_{z=0} - \frac{(r + \Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r}{\Delta r} + \rho g r \sin \theta = 0\]

Take the limit as \(\Delta r \to 0\).

\[-r \phi_{zz}|_{z=L} - \phi_{zz}|_{z=0} - \lim_{\Delta r \to 0} \frac{(r + \Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_{r}}{\Delta r} + \rho g r \sin \theta = 0\]

The second term is the definition of the first derivative of \(r\phi_{rz}\).

\[-r \phi_{zz}|_{z=L} - \phi_{zz}|_{z=0} - \frac{d}{dr}(r\phi_{rz}) + \rho g r \sin \theta = 0\]

Now substitute the expressions for \(\phi_{rz}\) and \(\phi_{zz}\).

\[\phi_{rz} = \tau_{rz} + \rho v_z^2 = \tau_{rz}\]

\[\phi_{zz} = p\delta_{zz} + \tau_{zz} + \rho v_z^2 v_z = p + \rho v_z^2\]

Since \(v_z\) does not depend on \(z\), the \(\rho v_z^2\) terms cancel.

\[-r \frac{p|_{z=L} + \rho v_z^2|_{z=L} - p|_{z=0} - \rho v_z^2|_{z=0}}{L} - \frac{d}{dr}(r\tau_{rz}) + \rho g r \sin \theta = 0\]

Make it so that \(\rho g r \sin \theta\) is in the fraction.

\[-r \frac{p|_{z=L} - p|_{z=0} - \rho g L \sin \theta}{L} - \frac{d}{dr}(r\tau_{rz}) = 0\]
From the schematic in Fig. 2B.8, we see that \( h = L \sin \theta \). It is here where \( \theta \) disappears from the equation; hence, it does not need to be measured.

\[
-r \frac{p|_{z=L} - p|_{z=0} - \rho gh}{L} - \frac{d}{dr}(r \tau_{rz}) = 0
\]

So we have

\[
\frac{d}{dr}(r \tau_{rz}) = \frac{\rho gh + (p|_{z=0} - p|_{z=L})}{L} r.
\]

It is thanks to the manometer underneath the capillary that the principles of fluid statics can be applied to compute the quantity in the numerator. Let

\[
\rho = \text{the density of water}
\]

\[
\rho_C = \text{the density of } \text{CCl}_4
\]

\[
H' = \text{the distance between points } B \text{ and } C \text{ in Fig. 2B.8}.
\]

\( D \) and \( E \) are at the same height in the \text{CCl}_4; thus, the pressures at these levels must be equal.

\[
\left( p|_{z=L} + \rho g H' + \rho_C g H \right)_{\text{pressure at } D} = \left( p|_{z=0} + \rho gh + \rho g H' + \rho g H \right)_{\text{pressure at } E}
\]

The \( \rho g H' \) terms cancel, so the distance between points \( B \) and \( C \) is not needed. Solve this equation for the quantity in the numerator

\[
\rho gh + p|_{z=0} - p|_{z=L} = (\rho_C - \rho) g H
\]

and substitute it into the differential equation.

\[
\frac{d}{dr}(r \tau_{rz}) = \frac{(\rho_C - \rho) g H}{L} r
\]

From Newton’s law of viscosity we know that \( \tau_{rz} = -\mu (dv_z/dr) \), so

\[
\frac{d}{dr} \left( -\mu r \frac{dv_z}{dr} \right) = \frac{(\rho_C - \rho) g H}{L} r.
\]

We thus have a differential equation for the velocity distribution in the capillary. The boundary conditions for it are obtained from the assumptions that the velocity is maximum furthest from the wall (at \( r = 0 \)) and that no slipping occurs between the fluid and the wall (at \( r = R \)).

\[
\text{B.C. 1 : } \frac{dv_z}{dr} = 0 \quad \text{when} \quad r = 0
\]

\[
\text{B.C. 2 : } v_z = 0 \quad \text{when} \quad r = R
\]

Integrate both sides of the differential equation with respect to \( r \).

\[
-\mu r \frac{dv_z}{dr} = \frac{(\rho_C - \rho) g H}{2L} r^2 + C_1
\]

Apply the first boundary condition now to determine \( C_1 \).

\[
-\mu(0) \frac{dv_z}{dr} \bigg|_{r=0} = \frac{(\rho_C - \rho) g H}{2L} (0)^2 + C_1 \quad \rightarrow \quad 0 = C_1
\]

www.stemjock.com
Divide both sides by \(-\mu r\).

\[ \frac{dv_z}{dr} = -\frac{(\rho C - \rho)gH}{2\mu L}r \]

Integrate both sides of the differential equation with respect to \(r\) once more.

\[ v_z(r) = -\frac{(\rho C - \rho)gH}{4\mu L}r^2 + C_2 \]

Apply the second boundary condition now to determine \(C_2\).

\[ v_z(R) = -\frac{(\rho C - \rho)gH}{4\mu L}R^2 + C_2 = 0 \quad \rightarrow \quad C_2 = \frac{(\rho C - \rho)gH}{4\mu L}R^2 \]

With the constants of integration in hand, the velocity distribution is known.

\[ v_z(r) = -\frac{(\rho C - \rho)gH}{4\mu L}r^2 + \frac{(\rho C - \rho)gH}{4\mu L}R^2 \]

\[ = \frac{(\rho C - \rho)gH}{4\mu L}(R^2 - r^2) \]

This result can be used to obtain the mass rate of flow \(w\).

\[ w = \frac{dm}{dt} = \frac{d}{dt}(\rho V) = \rho \frac{dV}{dt} \]

The volumetric flow \(dV/dt\) is equal to the average velocity in the capillary times its cross-sectional area.

\[ w = \rho \langle v_z \rangle \cdot \pi R^2 \]

The average velocity is found by integrating \(v_z\) over the area of the cross-section and then dividing by that area.

\[ w = \rho \left( \frac{1}{2\pi R^2} \int v_z dA \right) \cdot \pi R^2 \]

\[ = \rho \int_0^R v_z(2\pi r dr) \]

\[ = 2\pi \rho \int_0^R rv_z dr \]

\[ = 2\pi \rho \int_0^R r \left( \frac{(\rho C - \rho)gH}{4\mu L} \right) (R^2 - r^2) dr \]

\[ = \pi \rho \frac{(\rho C - \rho)gH}{2\mu L} \int_0^R (rR^2 - r^3) dr \]

\[ = \pi \rho \frac{(\rho C - \rho)gH}{2\mu L} \left( \frac{r^2R^2}{2} - \frac{r^4}{4} \right) \bigg|_0^R \]

\[ = \pi \rho \frac{(\rho C - \rho)gHR^4}{8\mu L} \]

www.stemjock.com
Use Eq. 1.1-3, \( \nu = \mu / \rho \), to write the density of water \( \rho \) in terms of the kinematic viscosity \( \nu \) and viscosity \( \mu \). Also, write the radius in terms of the diameter.

\[
w = \frac{\pi \left( \rho C - \frac{\mu}{\nu} \right) gH \left( \frac{D}{2} \right)^4}{8\nu L} \\
= \frac{\pi \left( \rho C \nu - \mu \right) gHD^4}{128\nu^2 L}
\]

Before we plug in the numbers, convert the units so that the desired units of \( \text{lb}_m/\text{hr} \) are obtained. \( \mu \) and \( \nu \) for water at 20°C are given on page 14 in Table 1.1-2. Other conversion factors are on page 868 and 870.

\[
\rho C = 1.594 \left( \frac{\text{g}}{\text{cm}^3} \right) \times \left( \frac{2.54 \text{ cm}}{1 \text{ in}} \right)^3 \times \frac{1 \text{ lb}_m}{1000 \text{ g}} \times \frac{2.2046 \text{ lb}_m}{1 \text{ lb}} \approx 0.0575863 \text{ lb}_m/\text{in}^3
\]

\[
\nu = 0.010037 \left( \frac{\text{cm}^2}{\text{s}} \right) \times \left( \frac{1 \text{ in}}{2.54 \text{ cm}} \right)^2 \times \frac{3600 \text{ s}}{1 \text{ hr}} \approx 5.60066 \text{ in}^2/\text{hr}
\]

\[
\mu = 1.0019 \left( \frac{\text{mPa} \cdot \text{s}}{1 \text{ mPa} \cdot \text{s}} \right) \times \frac{2.4191 \text{ lb}_m}{1 \text{ lb}_m/\text{hr}} \times \frac{1 \text{ ft}}{12 \text{ in}} \approx 0.201975 \text{ lb}_m/\text{in} \cdot \text{hr}
\]

\[
g = 9.81 \left( \frac{\text{m}}{1 \text{ m}} \right) \times \frac{3.28 \text{ ft}}{1 \text{ m}} \times \frac{12 \text{ in}}{1 \text{ ft}} \times \left( \frac{3600 \text{ s}}{1 \text{ hr}} \right)^2 \approx 5.00414 \times 10^9 \text{ in}^2/\text{hr}^2
\]

\( H = 1.0 \text{ in} \)
\( D = 0.010 \text{ in} \)
\( L = 120 \text{ in} \)

Therefore,

\[
w \approx \frac{\pi(0.0576 \cdot 5.6 - 0.202)(5 \times 10^9)(1.0)(0.010)^4 \text{ lb}_m \cdot \text{in}^5}{128(5.6)^2(120) \text{ in}^2/\text{hr}^2}
\]

\[
\approx 3.9 \times 10^{-5} \text{ lb}_m/\text{hr}.
\]