Problem 2C.4

Falling-cylinder viscometer (see Fig. 2C.4).[6] A falling-cylinder viscometer consists of a long vertical cylindrical container (radius R) capped at both ends, with a solid cylindrical slug (radius κR). The slug is equipped with fins so that its axis is coincident with that of the tube.

One can observe the rate of descent of the slug in the cylindrical container when the latter is filled with fluid. Find an equation that gives the viscosity of the fluid in terms of the terminal velocity v_0 of the slug and the various geometrical quantities shown in the figure.

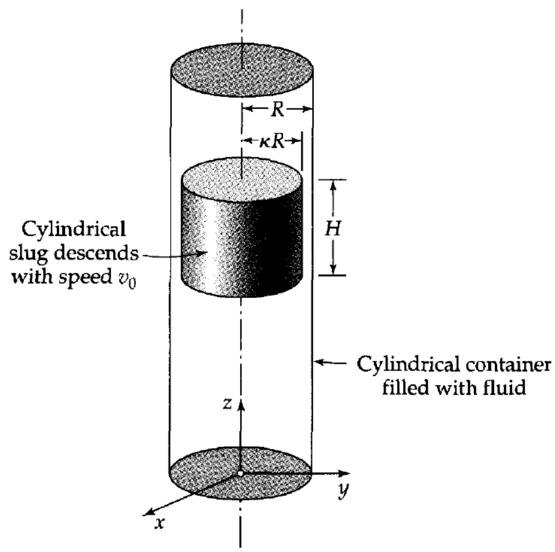

Fig. 2C.4 A falling-cylinder viscometer with a tightly fitting solid cylinder moving vertically. The cylinder is usually equipped with fins to maintain centering within the tube. The fluid completely fills the tube, and the top and bottom are closed.

(a) Show that the velocity distribution in the annular slit is given by

$$\frac{v_z}{v_0} = -\frac{(1-\xi^2) - (1+\kappa^2)\ln(1/\xi)}{(1-\kappa^2) - (1+\kappa^2)\ln(1/\kappa)} \tag{2C.4-1}$$

in which $\xi = r/R$ is a dimensionless radial coordinate.

(b) Make a force balance on the cylindrical slug and obtain

$$\mu = \frac{(\rho_0 - \rho)g(\kappa R)^2}{2v_0}\left[\left(\ln\frac{1}{\kappa}\right) - \left(\frac{1-\kappa^2}{1+\kappa^2}\right)\right] \tag{2C.4-2}$$

in which ρ and ρ_0 are the densities of the fluid and the slug, respectively.

(c) Show that, for small slit widths, the result in (b) may be expanded in powers of $\varepsilon = 1 - \kappa$ to give

$$\mu = \frac{(\rho_0 - \rho)gR^2\varepsilon^3}{6v_0}\left(1 - \frac{1}{2}\varepsilon - \frac{13}{20}\varepsilon^2 + \cdots\right) \tag{2C.4-3}$$

See §C.2 for information on expansions in Taylor series.

Solution

[6] J. Lohrenz, G. W. Swift, and F. Kurata, *AIChE Journal*, **6**, 547-550 (1960) and **7**, 6S (1961); E. Ashare, R. B. Bird, and J. A. Lescarboura, *AIChE Journal*, **11**, 910-916 (1965) F. J. Eichstadt and G. W. Swift, *AIChE Journal*, **12**, 1179-1183 (1966); M. C. S. Chen, J. A. Lescarboura, *AIChE Journal*, **14**, 123-127 (1968).

Part (a)

For this problem we choose a cylindrical coordinate system with the origin at the bottom of the slug's center. We assume that as the slug falls, the fluid in the annular slit flows in the z-direction and varies as a function of radius r.

$$v_z = v_z(r)$$

As a result, only ϕ_{rz} (the z-momentum in the positive r-direction) and ϕ_{zz} (the z-momentum in the positive z-direction) contribute to the momentum balance. The pressure is assumed to vary with height z.

$$p = p(z)$$

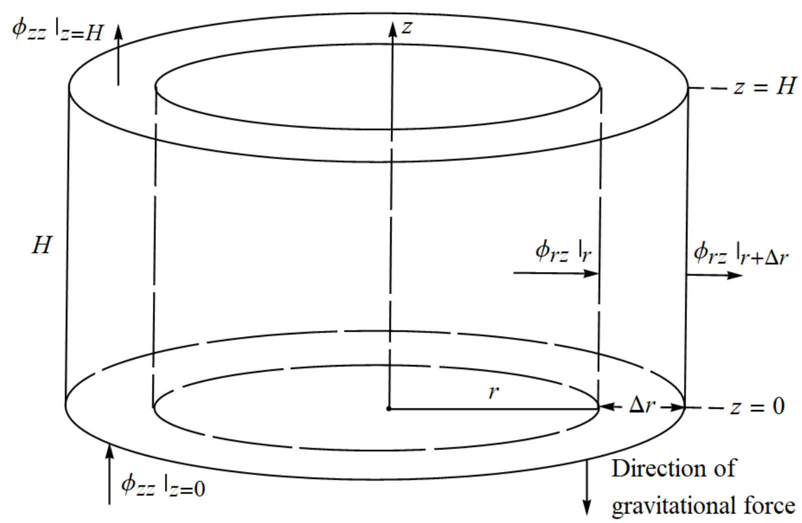

Figure 1: This is the shell over which the momentum balance is made for flow through an annular slit.

Rate of z-momentum into the shell at $z = 0$:	$(2\pi r\Delta r)\phi_{zz}\|_{z=0}$
Rate of z-momentum out of the shell at $z = H$:	$(2\pi r\Delta r)\phi_{zz}\|_{z=H}$
Rate of z-momentum into the shell at r:	$(2\pi r H)\phi_{rz}\|_{r}$
Rate of z-momentum out of the shell at $r + \Delta r$:	$[2\pi(r+\Delta r)H]\phi_{rz}\|_{r+\Delta r}$
Component of gravitational force on the shell in z-direction:	$-(2\pi r\Delta r H)\rho g$

If we assume steady flow, then the momentum balance is

$$\text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = \mathbf{0}.$$

Considering only the z-component, we have

$$(2\pi r\Delta r)\phi_{zz}|_{z=0} - (2\pi r\Delta r)\phi_{zz}|_{z=H} + (2\pi r H)\phi_{rz}|_{r} - [2\pi(r+\Delta r)H]\phi_{rz}|_{r+\Delta r} - (2\pi r\Delta r H)\rho g = 0.$$

Factor the left side.

$$-2\pi r\Delta r(\phi_{zz}|_{z=H} - \phi_{zz}|_{z=0}) - 2\pi H[(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_{r}] - 2\pi r\Delta r H\rho g = 0$$

Divide both sides by $-2\pi\Delta r H$.

$$r\frac{\phi_{zz}|_{z=H} - \phi_{zz}|_{z=0}}{H} + \frac{(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r}{\Delta r} + \rho g r = 0$$

Take the limit as $\Delta r \to 0$.

$$r\frac{\phi_{zz}|_{z=H} - \phi_{zz}|_{z=0}}{H} + \lim_{\Delta r\to 0}\frac{(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r}{\Delta r} + \rho g r = 0$$

The second term is the definition of the first derivative of $r\phi_{rz}$.

$$r\frac{\phi_{zz}|_{z=H} - \phi_{zz}|_{z=0}}{H} + \frac{d}{dr}(r\phi_{rz}) + \rho g r = 0$$

Now substitute the expressions for ϕ_{rz} and ϕ_{zz}.

$$\begin{aligned} \phi_{rz} &= \tau_{rz} + \cancel{\rho v_r v_z} = \tau_{rz} \\ \phi_{zz} &= p\delta_{zz} + \cancel{\tau_{zz}} + \rho v_z v_z = p(z) + \rho v_z^2 \end{aligned}$$

Since v_z does not depend on z, the ρv_z^2 terms cancel.

$$r\frac{p|_{z=H} + \cancel{\rho v_z^2|_{z=H}} - p|_{z=0} - \cancel{\rho v_z^2|_{z=0}}}{H} + \frac{d}{dr}(r\tau_{rz}) + \rho g r = 0$$

Make it so that $\rho g r$ is in the fraction.

$$r\frac{p|_{z=H} + \rho g H - p|_{z=0}}{H} + \frac{d}{dr}(r\tau_{rz}) = 0$$

Place $\rho g 0$ in the numerator.

$$r\frac{p|_{z=H} + \rho g H - p|_{z=0} - \rho g 0}{H} + \frac{d}{dr}(r\tau_{rz}) = 0$$

The point of doing this is that now we can use the modified pressure $\mathscr{P}_z = p(z) + \rho g z$.

$$r\frac{\mathscr{P}_H - \mathscr{P}_0}{H} + \frac{d}{dr}(r\tau_{rz}) = 0$$

So we have

$$\frac{d}{dr}(r\tau_{rz}) = -\frac{\mathscr{P}_H - \mathscr{P}_0}{H} r.$$

From Newton's law of viscosity we know that $\tau_{rz} = -\mu(dv_z/dr)$, so

$$\frac{d}{dr}\left(-\mu r\frac{dv_z}{dr}\right) = -\frac{\mathscr{P}_H - \mathscr{P}_0}{H} r.$$

The boundary conditions for this differential equation are obtained from assuming that no slip occurs between the fluid and the walls of the slug and container. That is, at $r = \kappa R$ the fluid travels with the slug ($v_z = -v_0$) and at $r = R$ the fluid is stationary ($v_z = 0$).

$$\begin{aligned} &\text{B.C. 1}: \quad v_z = -v_0, \quad \text{at } r = \kappa R \\ &\text{B.C. 2}: \quad v_z = 0, \qquad \text{at } r = R \end{aligned}$$

Integrate both sides of the differential equation with respect to r.

$$-\mu r \frac{dv_z}{dr} = -\frac{\mathscr{P}_H - \mathscr{P}_0}{2H} r^2 + C_1.$$

Divide both sides by $-\mu r$.

$$\frac{dv_z}{dr} = \frac{\mathscr{P}_H - \mathscr{P}_0}{2\mu H} r - \frac{C_1}{\mu r}$$

Integrate both sides of the differential equation with respect to r once more.

$$v_z(r) = \frac{\mathscr{P}_H - \mathscr{P}_0}{4\mu H} r^2 - \frac{C_1}{\mu} \ln r + C_2$$

Apply the boundary conditions here to determine C_1 and C_2.

$$v_z(\kappa R) = \frac{\mathscr{P}_H - \mathscr{P}_0}{4\mu H} (\kappa R)^2 - \frac{C_1}{\mu} \ln(\kappa R) + C_2 = -v_0$$
$$v_z(R) = \frac{\mathscr{P}_H - \mathscr{P}_0}{4\mu H} R^2 - \frac{C_1}{\mu} \ln R + C_2 = 0$$

Solving this system of equations, we get

$$C_1 = \frac{\mu}{\ln(1/\kappa)} \left[\frac{(\mathscr{P}_H - \mathscr{P}_0)R^2}{4\mu H}(1-\kappa^2) - v_0 \right]$$
$$C_2 = \frac{(\mathscr{P}_H - \mathscr{P}_0)R^2}{4\mu H} \left[(1-\kappa^2)\frac{\ln R}{\ln(1/\kappa)} - 1 \right] - \frac{\ln R}{\ln(1/\kappa)} v_0.$$

So we have for the velocity distribution,

$$v_z(r) = \frac{\mathscr{P}_H - \mathscr{P}_0}{4\mu H} r^2 - \frac{\ln r}{\ln(1/\kappa)} \left[\frac{(\mathscr{P}_H - \mathscr{P}_0)R^2}{4\mu H}(1-\kappa^2) - v_0 \right] + \frac{(\mathscr{P}_H - \mathscr{P}_0)R^2}{4\mu H} \left[(1-\kappa^2)\frac{\ln R}{\ln(1/\kappa)} - 1 \right] - \frac{\ln R}{\ln(1/\kappa)} v_0.$$

Factor the right side.

$$v_z(r) = \frac{\mathscr{P}_H - \mathscr{P}_0}{4\mu H}(r^2 - R^2) + \frac{(\mathscr{P}_H - \mathscr{P}_0)R^2}{4\mu H} \frac{1-\kappa^2}{\ln(1/\kappa)}(\ln R - \ln r) + \frac{v_0}{\ln(1/\kappa)}(\ln r - \ln R)$$

Thus,

$$v_z(r) = \frac{(\mathscr{P}_H - \mathscr{P}_0)R^2}{4\mu H} \left[\left(\frac{r}{R}\right)^2 - 1 + \frac{1-\kappa^2}{\ln(1/\kappa)} \ln \frac{R}{r} \right] - \frac{v_0}{\ln(1/\kappa)} \ln \frac{R}{r}.$$

Here we introduce the dimensionless radial coordinate $\xi = r/R$.

$$v_z(\xi) = \frac{(\mathscr{P}_H - \mathscr{P}_0)R^2}{4\mu H} \left[\xi^2 - 1 + \frac{1-\kappa^2}{\ln(1/\kappa)} \ln(1/\xi) \right] - \frac{v_0}{\ln(1/\kappa)} \ln(1/\xi) \tag{1}$$

Our aim now is to eliminate the coefficient of the square brackets because it's not in the desired answer. As the slug descends in the container, it displaces a certain volume of fluid per unit time. That same volume per unit time must be what flows up the side of the slug in the annular slit. The following relation can be written from this.

$$\left.\frac{dV}{dt}\right|_{\text{displaced from bottom of slug}} = \left.\frac{dV}{dt}\right|_{\text{up side of slug in annular slit}}$$

The volumetric flow rate is velocity times area, so on the left side it's just $v_0 \cdot \pi(\kappa R)^2$. Since the velocity varies radially in the slit, we will have to integrate the velocity over the area on the right side.

$$\begin{aligned}
v_0 \cdot \pi\kappa^2 R^2 &= \int v_z \, dA \\
v_0 \cdot \pi\kappa^2 R^2 &= \int_{\kappa R}^{R} v_z (2\pi r \, dr) \\
v_0 \cdot \pi\kappa^2 R^2 &= 2\pi \int_{\kappa R}^{R} r v_z \, dr
\end{aligned}$$

$$\frac{v_0 \kappa^2 R^2}{2} = \int_{\kappa R}^{R} r \left\{ \frac{(\mathscr{P}_H - \mathscr{P}_0) R^2}{4\mu H} \left[\left(\frac{r}{R}\right)^2 - 1 + \frac{1-\kappa^2}{\ln(1/\kappa)} \ln\frac{R}{r} \right] - \frac{v_0}{\ln(1/\kappa)} \ln\frac{R}{r} \right\} dr$$

Make the substitution,

$$\begin{aligned}
\xi &= \frac{r}{R} \quad \rightarrow \quad R\xi = r \\
d\xi &= \frac{dr}{R} \quad \rightarrow \quad R\, d\xi = dr,
\end{aligned}$$

to get

$$\begin{aligned}
\frac{v_0 \kappa^2 R^2}{2} &= \int_{\kappa}^{1} (R\xi) \left\{ \frac{(\mathscr{P}_H - \mathscr{P}_0) R^2}{4\mu H} \left[\xi^2 - 1 + \frac{1-\kappa^2}{\ln(1/\kappa)} \ln(1/\xi) \right] - \frac{v_0}{\ln(1/\kappa)} \ln(1/\xi) \right\} (R\, d\xi) \\
\frac{v_0 \kappa^2 R^2}{2} &= R^2 \left\{ \frac{(\mathscr{P}_H - \mathscr{P}_0) R^2}{4\mu H} \int_{\kappa}^{1} \left[\xi^3 - \xi + \frac{1-\kappa^2}{\ln(1/\kappa)} \xi \ln(1/\xi) \right] d\xi - \frac{v_0}{\ln(1/\kappa)} \int_{\kappa}^{1} \xi \ln(1/\xi) \, d\xi \right\} \\
\frac{v_0 \kappa^2}{2} &= \frac{(\mathscr{P}_H - \mathscr{P}_0) R^2}{4\mu H} \int_{\kappa}^{1} \left[\xi^3 - \xi - \frac{1-\kappa^2}{\ln(1/\kappa)} \xi \ln \xi \right] d\xi + \frac{v_0}{\ln(1/\kappa)} \int_{\kappa}^{1} \xi \ln \xi \, d\xi
\end{aligned}$$

Bring the last term on the right side over to the left.

$$\begin{aligned}
\frac{v_0 \kappa^2}{2} - \frac{v_0}{\ln(1/\kappa)} \int_{\kappa}^{1} \xi \ln \xi \, d\xi &= \frac{(\mathscr{P}_H - \mathscr{P}_0) R^2}{4\mu H} \int_{\kappa}^{1} \left[\xi^3 - \xi - \frac{1-\kappa^2}{\ln(1/\kappa)} \xi \ln \xi \right] d\xi \\
\frac{v_0 \kappa^2}{2} - \frac{v_0}{\ln(1/\kappa)} \left(-\frac{\xi^2}{4} + \frac{\xi^2}{2} \ln \xi \right) \Big|_{\kappa}^{1} &= \frac{(\mathscr{P}_H - \mathscr{P}_0) R^2}{4\mu H} \left[\frac{\xi^4}{4} - \frac{\xi^2}{2} - \frac{1-\kappa^2}{\ln(1/\kappa)} \left(-\frac{\xi^2}{4} + \frac{\xi^2}{2} \ln \xi \right) \right] \Big|_{\kappa}^{1} \\
\cancel{\frac{v_0 \kappa^2}{2}} - \frac{v_0}{4\ln(1/\kappa)} (-1 + \kappa^2 - \cancel{2\kappa^2 \ln \kappa}) &= \frac{(\mathscr{P}_H - \mathscr{P}_0) R^2}{4\mu H} \left[-\frac{1}{4} + \frac{\kappa^2}{2} - \frac{\kappa^4}{4} - \frac{1-\kappa^2}{4\ln(1/\kappa)} (-1 + \kappa^2 - 2\kappa^2 \ln \kappa) \right] \\
\frac{v_0}{4\ln(1/\kappa)} (1 - \kappa^2) &= \frac{(\mathscr{P}_H - \mathscr{P}_0) R^2}{4\mu H} \left[-\frac{1}{4} + \frac{\kappa^4}{4} + \frac{(1-\kappa^2)^2}{4\ln(1/\kappa)} \right] \\
\frac{v_0}{\ln(1/\kappa)} (1 - \kappa^2) &= \frac{(\mathscr{P}_H - \mathscr{P}_0) R^2}{4\mu H} \left[\kappa^4 - 1 + \frac{(1-\kappa^2)^2}{\ln(1/\kappa)} \right] \\
-\frac{v_0}{\ln(1/\kappa)} (\kappa^2 - 1) &= \frac{(\mathscr{P}_H - \mathscr{P}_0) R^2}{4\mu H} \left[(\kappa^2 + 1)(\kappa^2 - 1) + \frac{(\kappa^2 - 1)^2}{\ln(1/\kappa)} \right] \\
-\frac{v_0}{\ln(1/\kappa)} &= \frac{(\mathscr{P}_H - \mathscr{P}_0) R^2}{4\mu H} \left[(\kappa^2 + 1) + \frac{\kappa^2 - 1}{\ln(1/\kappa)} \right] \\
-\frac{v_0}{\ln(1/\kappa)} &= \frac{(\mathscr{P}_H - \mathscr{P}_0) R^2}{4\mu H} \left[\frac{(\kappa^2 + 1)\ln(1/\kappa) + \kappa^2 - 1}{\ln(1/\kappa)} \right]
\end{aligned}$$

Consequently,

$$\frac{(\mathscr{P}_H - \mathscr{P}_0)R^2}{4\mu H} = -\frac{v_0}{(\kappa^2 + 1)\ln(1/\kappa) + \kappa^2 - 1}. \tag{2}$$

Substitute this result into equation (1).

$$v_z(\xi) = -\frac{v_0}{(\kappa^2+1)\ln(1/\kappa) + \kappa^2 - 1}\left[\xi^2 - 1 + \frac{1-\kappa^2}{\ln(1/\kappa)}\ln(1/\xi)\right] - \frac{v_0}{\ln(1/\kappa)}\ln(1/\xi)$$

Divide both sides by v_0 and factor the minus signs.

$$\begin{aligned}
\frac{v_z}{v_0} &= -\left[\frac{\xi^2 - 1 + \frac{1-\kappa^2}{\ln(1/\kappa)}\ln(1/\xi)}{(\kappa^2+1)\ln(1/\kappa) + (\kappa^2-1)} + \frac{\ln(1/\xi)}{\ln(1/\kappa)}\right] \\
&= -\left\{\frac{(\xi^2-1)\ln(1/\kappa) + (1-\kappa^2)\ln(1/\xi)}{\ln(1/\kappa)[(\kappa^2+1)\ln(1/\kappa) + (\kappa^2-1)]} + \frac{(\kappa^2+1)\ln(1/\kappa)\ln(1/\xi) + (\kappa^2-1)\ln(1/\xi)}{\ln(1/\kappa)[(\kappa^2+1)\ln(1/\kappa) + (\kappa^2-1)]}\right\} \\
&= -\left\{\frac{(\xi^2-1)\ln(1/\kappa) + \cancel{(1-\kappa^2)\ln(1/\xi)} + (\kappa^2+1)\ln(1/\kappa)\ln(1/\xi) + \cancel{(\kappa^2-1)\ln(1/\xi)}}{\ln(1/\kappa)[(\kappa^2+1)\ln(1/\kappa) + (\kappa^2-1)]}\right\} \\
&= -\left\{\frac{(\xi^2-1)\cancel{\ln(1/\kappa)} + (\kappa^2+1)\cancel{\ln(1/\kappa)}\ln(1/\xi)}{\cancel{\ln(1/\kappa)}[(\kappa^2+1)\ln(1/\kappa) + (\kappa^2-1)]}\right\} \\
&= -\left[-\frac{(1-\xi^2) - (\kappa^2+1)\ln(1/\xi)}{(\kappa^2+1)\ln(1/\kappa) + (\kappa^2-1)}\right] \\
&= -\left[\frac{(1-\xi^2) - (\kappa^2+1)\ln(1/\xi)}{-(\kappa^2+1)\ln(1/\kappa) - (\kappa^2-1)}\right]
\end{aligned}$$

Therefore,

$$\frac{v_z}{v_0} = -\frac{(1-\xi^2) - (1+\kappa^2)\ln(1/\xi)}{(1-\kappa^2) - (1+\kappa^2)\ln(1/\kappa)},$$

where $\xi = r/R$.

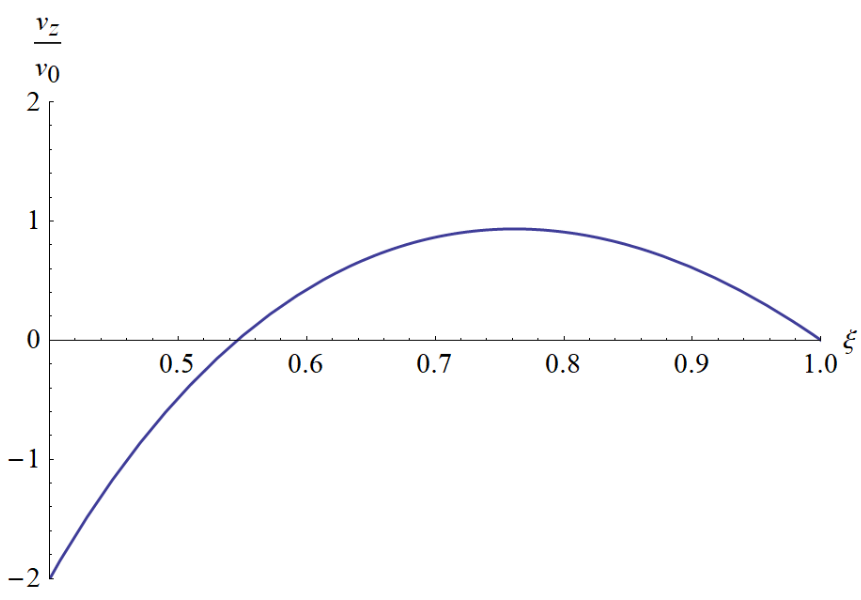

Figure 2: This is a plot of the velocity distribution when $\kappa = 0.4$ and $v_0 = 2$ for $\kappa \le \xi \le 1$.

Part (b)

To determine the viscosity, the sum of the forces in the z-direction will be considered. There are five forces that need to be taken into account in the free body diagram of the slug: (1) the gravitational force, (2) the buoyant force, (3) the viscous force acting on the side from the flow of fluid in the annulus, (4) the weight of the water (pressure) acting over the top of the slug's surface, and (5) the weight of the water (pressure) acting over the bottom of the slug's surface. The forces due to pressure are normal to the slug's surface, and the viscous force is parallel to the slug's surface in the direction opposing the slug's motion.

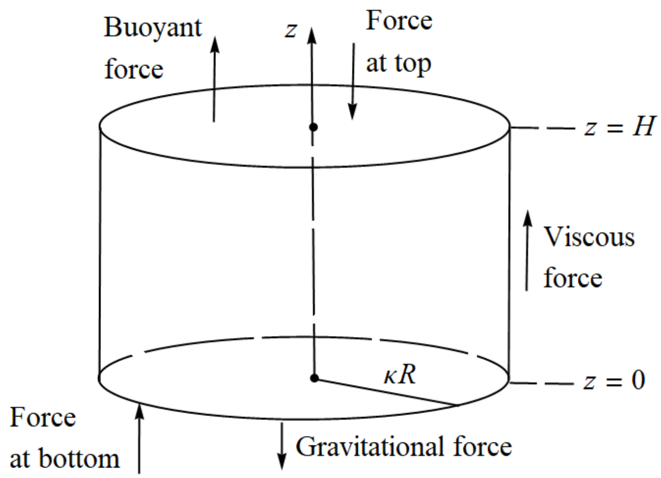

Figure 3: This is the free body diagram of the slug.

$$\text{Gravitational force} = \text{Density of slug} \times \text{Volume of slug} \times \text{Acceleration} = \rho_0 \times \pi(\kappa R)^2 H \times g$$

$$\text{Buoyant force} = \text{Density of fluid} \times \text{Volume of slug} \times \text{Acceleration} = \rho \times \pi(\kappa R)^2 H \times g$$

$$\text{Viscous force} = \text{Shearing stress} \times \text{Surface area} = (-\tau_{rz}|_{r=\kappa R}) \times 2\pi\kappa R H$$

$$\text{Force at bottom} = \text{Pressure at bottom} \times \text{Surface area} = \mathscr{P}_0 \times \pi(\kappa R)^2$$

$$\text{Force at top} = \text{Pressure at top} \times \text{Surface area} = \mathscr{P}_H \times \pi(\kappa R)^2$$

The minus sign in the shearing stress is due to the fact that the fluid in the annulus is at a higher radius acting on the slug's surface, which has a lower radius. Use Newton's law of viscosity, $\tau_{rz} = -\mu(dv_z/dr)$, to evaluate it.

$$\begin{aligned}
\text{Viscous force} &= (-\tau_{rz}|_{r=\kappa R}) \times 2\pi\kappa R H \\
&= -\left(-\mu \frac{dv_z}{dr}\right)\bigg|_{r=\kappa R} \times 2\pi\kappa R H \\
&= \frac{dv_z}{dr}\bigg|_{r=\kappa R} \times 2\pi\kappa R H \mu \\
&= -\frac{v_0}{\kappa R} \frac{1-\kappa^2}{1-\kappa^2-(1+\kappa^2)\ln(1/\kappa)} \times 2\pi\kappa R H \mu \\
&= -\frac{1-\kappa^2}{1-\kappa^2-(1+\kappa^2)\ln(1/\kappa)} \times 2\pi H \mu v_0
\end{aligned}$$

Because the slug is falling at terminal velocity, the acceleration is zero, so the sum of the forces in the z-direction is equal to 0.

$$\sum F_z = ma_z = 0$$

Plug in the forces on the left side. The positive z-direction is chosen to be upward, so forces pointing up are positive and those pointing down are negative.

$$-\rho_0\pi(\kappa R)^2 Hg + \rho\pi(\kappa R)^2 Hg - \frac{1-\kappa^2}{1-\kappa^2-(1+\kappa^2)\ln(1/\kappa)} \times 2\pi H\mu v_0 + \mathscr{P}_0 \times \pi(\kappa R)^2 - \mathscr{P}_H \times \pi(\kappa R)^2 = 0$$

Divide both sides by $\pi(\kappa R)^2 H$.

$$(\rho-\rho_0)g - \frac{1-\kappa^2}{1-\kappa^2-(1+\kappa^2)\ln(1/\kappa)} \times \frac{2\mu v_0}{\kappa^2 R^2} + \frac{\mathscr{P}_0-\mathscr{P}_H}{H} = 0$$

Solve equation (2) for $(\mathscr{P}_0 - \mathscr{P}_H)/H$ and substitute the expression here.

$$(\rho-\rho_0)g - \frac{1-\kappa^2}{1-\kappa^2-(1+\kappa^2)\ln(1/\kappa)} \times \frac{2\mu v_0}{\kappa^2 R^2} + \frac{v_0}{(\kappa^2+1)\ln(1/\kappa)+\kappa^2-1} \times \frac{4\mu}{R^2} = 0$$

All that's left to do is to solve for μ.

$$\begin{aligned}
\kappa^2 R^2(\rho-\rho_0)g - \frac{2\mu v_0(1-\kappa^2)}{1-\kappa^2-(1+\kappa^2)\ln(1/\kappa)} - \frac{4\mu v_0 \kappa^2}{1-\kappa^2-(1+\kappa^2)\ln(1/\kappa)} &= 0 \\
\kappa^2 R^2(\rho-\rho_0)g - \frac{2\mu v_0(1+\kappa^2)}{1-\kappa^2-(1+\kappa^2)\ln(1/\kappa)} &= 0 \\
\kappa^2 R^2(\rho-\rho_0)g(1-\kappa^2) - \kappa^2 R^2(\rho-\rho_0)g(1+\kappa^2)\ln(1/\kappa) - 2\mu v_0(1+\kappa^2) &= 0 \\
\kappa^2 R^2(\rho-\rho_0)g[(1-\kappa^2)-(1+\kappa^2)\ln(1/\kappa)] &= 2\mu v_0(1+\kappa^2) \\
\frac{\kappa^2 R^2(\rho-\rho_0)g}{2v_0}\left[\frac{(1-\kappa^2)}{1+\kappa^2} - \ln(1/\kappa)\right] &= \mu
\end{aligned}$$

Therefore,

$$\mu = \frac{(\rho_0-\rho)g(\kappa R)^2}{2v_0}\left[\left(\ln\frac{1}{\kappa}\right) - \left(\frac{1-\kappa^2}{1+\kappa^2}\right)\right].$$

Part (c)

For small slit widths, κ is just barely less than one: $\kappa = 1-\varepsilon$, where $0 < \varepsilon \ll 1$. Substitute this into the result of part (b).

$$\begin{aligned}
\mu &= \frac{(\rho_0-\rho)g(\kappa R)^2}{2v_0}\left[\left(\ln\frac{1}{\kappa}\right) - \left(\frac{1-\kappa^2}{1+\kappa^2}\right)\right] \\
&= \frac{(\rho_0-\rho)g(1-\varepsilon)^2 R^2}{2v_0}\left[\ln\frac{1}{1-\varepsilon} - \frac{1-(1-\varepsilon)^2}{1+(1-\varepsilon)^2}\right] \\
&= \frac{(\rho_0-\rho)gR^2}{2v_0}(1-\varepsilon)^2\left[-\ln(1-\varepsilon) - \frac{2\varepsilon-\varepsilon^2}{2-2\varepsilon+\varepsilon^2}\right]
\end{aligned}$$

The Taylor series expansion for $\ln(1-\varepsilon)$ is

$$\ln(1-\varepsilon) = -\varepsilon - \frac{1}{2}\varepsilon^2 - \frac{1}{3}\varepsilon^3 - \frac{1}{4}\varepsilon^4 - \frac{1}{5}\varepsilon^5 - \cdots.$$

Plug this in to the formula for μ.

$$\begin{aligned}
\mu &= \frac{(\rho_0 - \rho)gR^2}{2v_0}(1-\varepsilon)^2 \left[-\left(-\varepsilon - \frac{1}{2}\varepsilon^2 - \frac{1}{3}\varepsilon^3 - \frac{1}{4}\varepsilon^4 - \frac{1}{5}\varepsilon^5 - \cdots\right) - \frac{2\varepsilon - \varepsilon^2}{2 - 2\varepsilon + \varepsilon^2}\right] \\
&= \frac{(\rho_0 - \rho)gR^2}{2v_0}(1-2\varepsilon+\varepsilon^2) \left[\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \frac{1}{4}\varepsilon^4 + \frac{1}{5}\varepsilon^5 + \cdots - \frac{2\varepsilon - \varepsilon^2}{2 - 2\varepsilon + \varepsilon^2}\right] \\
&= \frac{(\rho_0 - \rho)gR^2}{2v_0} \left[\left(\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \frac{1}{4}\varepsilon^4 + \frac{1}{5}\varepsilon^5 + \cdots\right)(1-2\varepsilon+\varepsilon^2) - \frac{2\varepsilon - \varepsilon^2}{2 - 2\varepsilon + \varepsilon^2}(1-2\varepsilon+\varepsilon^2)\right]
\end{aligned}$$

Use long division to obtain a series for the quotient.

$$\begin{array}{rl}
 & \varepsilon + \frac{1}{2}\varepsilon^2 - \frac{1}{4}\varepsilon^4 - \frac{1}{4}\varepsilon^5 + \cdots \\
2 - 2\varepsilon + \varepsilon^2 \big) & 2\varepsilon - \varepsilon^2 + 0\varepsilon^3 \\
(-) & 2\varepsilon - 2\varepsilon^2 + \varepsilon^3 \\
 & \varepsilon^2 - \varepsilon^3 \\
(-) & \varepsilon^2 - \varepsilon^3 + \frac{1}{2}\varepsilon^4 \\
 & -\frac{1}{2}\varepsilon^4 \\
(-) & -\frac{1}{2}\varepsilon^4 + \frac{1}{2}\varepsilon^5 - \frac{1}{4}\varepsilon^6 \\
 & -\frac{1}{2}\varepsilon^5 + \frac{1}{4}\varepsilon^6
\end{array}$$

Having terms up to ε^5 will suffice.

$$\begin{aligned}
\mu = \frac{(\rho_0 - \rho)gR^2}{2v_0} \bigg[&\left(\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \frac{1}{4}\varepsilon^4 + \frac{1}{5}\varepsilon^5 + \cdots\right)(1-2\varepsilon+\varepsilon^2) \\
&- \left(\varepsilon + \frac{1}{2}\varepsilon^2 - \frac{1}{4}\varepsilon^4 - \frac{1}{4}\varepsilon^5 + \cdots\right)(1-2\varepsilon+\varepsilon^2)\bigg].
\end{aligned}$$

Multiply the series together.

$$\begin{aligned}
&= \frac{(\rho_0 - \rho)gR^2}{2v_0} \bigg[\varepsilon + \varepsilon^2\left(-2+\frac{1}{2}\right) + \varepsilon^3\left(1 - 1 + \frac{1}{3}\right) + \varepsilon^4\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \varepsilon^5\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) + \cdots \\
&\qquad -\varepsilon - \varepsilon^2\left(-2+\frac{1}{2}\right) - \varepsilon^3(1-1) - \varepsilon^4\left(\frac{1}{2} - \frac{1}{4}\right) - \varepsilon^5\left(\frac{1}{2} - \frac{1}{4}\right)\bigg] \\
&= \frac{(\rho_0 - \rho)gR^2}{2v_0}\left(\frac{1}{3}\varepsilon^3 + \frac{1}{12}\varepsilon^4 + \frac{1}{30}\varepsilon^5 - \frac{1}{4}\varepsilon^4 - \frac{1}{4}\varepsilon^5 + \cdots\right)
\end{aligned}$$

Combine like-terms.

$$= \frac{(\rho_0 - \rho)gR^2}{2v_0}\left(\frac{1}{3}\varepsilon^3 - \frac{1}{6}\varepsilon^4 - \frac{13}{60}\varepsilon^5 + \cdots\right)$$

Factor $\varepsilon^3/3$ to obtain the final result. Therefore,

$$\mu = \frac{(\rho_0 - \rho)gR^2\varepsilon^3}{6v_0}\left(1 - \frac{1}{2}\varepsilon - \frac{13}{20}\varepsilon^2 + \cdots\right).$$