

Problem 3A.3

Effect of altitude on air pressure. When standing at the mouth of the Ontonagon River on the south shore of Lake Superior (602 ft above mean sea level), your portable barometer indicates a pressure of 750 mm Hg. Use the equation of motion to estimate the barometric pressure at the top of Government Peak (2023 ft above mean sea level) in the nearby Porcupine Mountains. Assume that the temperature at lake level is 70°F and that the temperature decreases with increasing altitude at a steady rate of 3°F per 1000 feet. The gravitational acceleration at the south shore of Lake Superior is about 32.19 ft/s², and its variation with altitude may be neglected in this problem.

Answer: 713 mm Hg = 9.49×10^4 N/m² (if $\rho = \rho(p)$)

Solution

Assuming that the air has constant density and viscosity, the equation of motion simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t} \rho \mathbf{v} + \nabla \cdot \rho \mathbf{v} \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

As the air is static, only two terms in the equation remain.

$$0 = -\nabla p + \rho \mathbf{g}$$

This is a vector equation, so it actually represents three scalar equations—one for each variable in the chosen coordinate system. Here a Cartesian coordinate system will be used, where the z -axis points opposite the direction of gravity.

$$\begin{aligned} \frac{\partial p}{\partial x} &= 0 \\ \frac{\partial p}{\partial y} &= 0 \\ \frac{\partial p}{\partial z} &= -\rho g_z \end{aligned}$$

The first two equations imply that the pressure is only a function of z .

$$\frac{dp}{dz} = -\rho g_z \quad (1)$$

Assuming that the air is an ideal gas, its equation of state is

$$pV = nRT.$$

Write n in terms of the molar mass and solve for the density ρ .

$$\begin{aligned} pV &= \frac{m}{M} RT \\ pM &= \frac{m}{V} RT \\ pM &= \rho RT \\ \rho &= \frac{pM}{RT} \end{aligned}$$

As a result, equation (1) becomes

$$\frac{dp}{dz} = -\frac{pMg_z}{RT}. \quad (2)$$

The temperature is not constant: it decreases with increasing altitude at a steady rate of 3°F per 1000 feet.

$$\frac{dT_F}{dz} = -\frac{3}{1000}$$

Integrate both sides with respect to z .

$$T_F(z) = -0.003z + C_1$$

Use the fact that the temperature is 70°F at 602 feet to determine C_1 .

$$T_F(602) = -0.003(602) + C_1 = 70 \quad \rightarrow \quad C_1 = 71.806$$

So then

$$T_F(z) = 71.806 - 0.003z.$$

The temperature used in the ideal gas law has to be absolute, so we change the temperature scale to Rankine, which increases at the same rate as Fahrenheit (an increase of 1°F corresponds to an increase of 1 R).

$$\begin{aligned} T_R(z) &= T_F(z) + 459.67 \\ &= 531.476 - 0.003z \end{aligned}$$

With this formula for the temperature equation (2) becomes

$$\begin{aligned} \frac{dp}{dz} &= -\frac{pMg_z}{R(531.476 - 0.003z)} \\ &= \frac{pMg_z}{R(0.003z - 531.476)}. \end{aligned}$$

This differential equation can be solved by separating variables.

$$\frac{dp}{p} = \frac{Mg_z}{R(0.003z - 531.476)} dz$$

Integrate both sides.

$$\begin{aligned} \int_{p_1}^{p_2} \frac{dp}{p} &= \int_{z_1}^{z_2} \frac{Mg_z}{R(0.003z - 531.476)} dz \\ \ln p \Big|_{p_1}^{p_2} &= \frac{Mg_z}{0.003R} \ln(0.003z - 531.476) \Big|_{z_1}^{z_2} \\ \ln \frac{p_2}{p_1} &= \frac{Mg_z}{0.003R} \ln \frac{0.003z_2 - 531.476}{0.003z_1 - 531.476} \\ \ln \frac{p_2}{p_1} &= \ln \left(\frac{0.003z_2 - 531.476}{0.003z_1 - 531.476} \right)^{Mg_z/(0.003R)} \end{aligned}$$

Exponentiate both sides and solve for p_2 .

$$\begin{aligned} \frac{p_2}{p_1} &= \left(\frac{0.003z_2 - 531.476}{0.003z_1 - 531.476} \right)^{Mg_z/(0.003R)} \\ p_2 &= p_1 \left(\frac{0.003z_2 - 531.476}{0.003z_1 - 531.476} \right)^{Mg_z/(0.003R)} \end{aligned}$$

To get an estimate for the molar mass of air, we assume the atmosphere is 80% nitrogen gas and 20% oxygen gas.

$$M = \left(2 \times 14.01 \frac{\text{lb}_m}{\text{lb-mol}} \right) 0.8 + \left(2 \times 16 \frac{\text{lb}_m}{\text{lb-mol}} \right) 0.2 = 28.816 \frac{\text{lb}_m}{\text{lb-mol}}$$

$$g_z = 32.19 \frac{\text{ft}}{\text{s}^2}$$

$$R = 4.9686 \times 10^4 \frac{\text{lb}_m \cdot \frac{\text{ft}^2}{\text{s}^2}}{\text{lb-mol} \cdot \text{R}}$$

This value of R was found in Appendix F on page 867. At Ontonagon River $z_1 = 602$ ft and $p_1 = 750$ mm Hg, and at Government's Peak $z_2 = 2023$ ft and p_2 is unknown. Plugging these numbers into the formula for p_2 gives

$$p_2 \approx 713 \text{ mm Hg} \times \frac{101325 \text{ N/m}^2}{760 \text{ mm Hg}} \approx 9.51 \times 10^4 \frac{\text{N}}{\text{m}^2}.$$