Problem 3B.15

Shape of free surface in tangential annular flow.

(a) A liquid is in the annular space between two vertical cylinders of radii κR and R, and the liquid is open to the atmosphere at the top. Show that when the inner cylinder rotates with an angular velocity Ω_i, and the outer cylinder is fixed, the free liquid surface has the shape

$$z_R - z = \frac{1}{2g}\left(\frac{\kappa^2 R\Omega_i}{1-\kappa^2}\right)^2 (\xi^{-2} + 4\ln\xi - \xi^2) \tag{3B.15-1}$$

in which z_R is the height of the liquid at the outer-cylinder wall, and $\xi = r/R$.

(b) Repeat (a) but with the inner cylinder fixed and the outer cylinder rotating with an angular velocity Ω_o. Show that the shape of the liquid surface is

$$z_R - z = \frac{1}{2g}\left(\frac{\kappa^2 R\Omega_o}{1-\kappa^2}\right)^2 [(\xi^{-2} - 1) + 4\kappa^{-2}\ln\xi - \kappa^{-4}(\xi^2 - 1)] \tag{3B.15-2}$$

(c) Draw a sketch comparing these two liquid-surface shapes.

Solution

For two concentric cylinders that rotate about their axis of symmetry, we assume that the fluid between them flows only in the θ-direction and that its velocity varies as a function of radius only.

$$\mathbf{v} = v_\theta(r)\hat{\boldsymbol{\theta}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $r = \kappa R$ and $r = R$. The tangential velocity is obtained by multiplying the angular velocity by the distance from the axis of rotation (the moment arm).

Boundary Condition 1: $\quad v_\theta(\kappa R) = \Omega_i \kappa R$

Boundary Condition 2: $\quad v_\theta(R) = \Omega_o R$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using cylindrical coordinates is the appropriate choice for this problem, so equations (1) and (2) will be used in (r, θ, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\bigg(\underbrace{\frac{\partial v_r}{\partial t}}_{=0}+\underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0}+\underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0}+\underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0}-\frac{v_\theta^2}{r}\bigg)=-\frac{\partial p}{\partial r}+\mu\bigg[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0}+\underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0}+\underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0}-\underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\bigg]+\underbrace{\rho g_r}_{=0}$$

$$\rho\bigg(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0}+\underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0}+\underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0}+\underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0}+\underbrace{\frac{v_r v_\theta}{r}}_{=0}\bigg)=\underbrace{-\frac{1}{r}\frac{\partial p}{\partial \theta}}_{=0}+\mu\bigg[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0}+\underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0}+\underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\bigg]+\underbrace{\rho g_\theta}_{=0}$$

$$\rho\bigg(\underbrace{\frac{\partial v_z}{\partial t}}_{=0}+\underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0}+\underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0}+\underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\bigg)=-\frac{\partial p}{\partial z}+\mu\bigg[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0}+\underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0}+\underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\bigg]+\rho g_z$$

The relevant equation for the velocity is the θ-equation, which has simplified considerably from the assumption that $\mathbf{v}=v_\theta(r)\hat{\boldsymbol{\theta}}$.

$$0=\mu\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right)$$

Divide both sides by μ.

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right)=0$$

Integrate both sides with respect to r.

$$\frac{1}{r}\frac{d}{dr}(rv_\theta)=C_1$$

Multiply both sides by r.

$$\frac{d}{dr}(rv_\theta)=C_1 r$$

Integrate both sides with respect to r once more.

$$rv_\theta=C_1\frac{r^2}{2}+C_2$$

Divide both sides by r to obtain the general solution.

$$v_\theta(r)=C_1\frac{r}{2}+\frac{C_2}{r}$$

Part (a)

Apply the two boundary conditions here to determine C_1 and C_2 for the case that only the inner cylinder is rotating.

$$v_\theta(\kappa R)=C_1\frac{\kappa R}{2}+\frac{C_2}{\kappa R}=\Omega_i\kappa R$$
$$v_\theta(R)=C_1\frac{R}{2}+\frac{C_2}{R}=0$$

Solving the system of equations yields

$$C_1=-\frac{2\kappa^2}{1-\kappa^2}\Omega_i \quad \text{and} \quad C_2=\frac{\kappa^2}{1-\kappa^2}R^2\Omega_i,$$

which means

$$\begin{aligned}
v_\theta(r) &= \left(-\frac{2\kappa^2}{1-\kappa^2}\Omega_i\right)\frac{r}{2} + \left(\frac{\kappa^2}{1-\kappa^2}R^2\Omega_i\right)\frac{1}{r} \\
&= \frac{\kappa^2}{1-\kappa^2}\left(-\frac{r}{R} + \frac{R}{r}\right)R\Omega_i \\
&= \frac{\kappa^2}{1-\kappa^2}\left(\frac{R^2 - r^2}{rR}\right)R\Omega_i \\
&= \frac{\kappa^2}{1-\kappa^2}\left(\frac{R^2 - r^2}{r}\right)\Omega_i.
\end{aligned}$$

Now that we have the velocity profile, we can use the r- and z-equations of motion to determine the pressure distribution.

$$\begin{aligned}
-\rho\frac{v_\theta^2}{r} &= -\frac{\partial p}{\partial r} \\
0 &= -\frac{\partial p}{\partial z} + \rho g_z
\end{aligned}$$

The z-axis points up the cylinders' axis of symmetry, which is opposite the direction of gravity: $\mathbf{g} = -g\hat{\mathbf{z}}$.

$$\frac{\partial p}{\partial r} = \rho\frac{v_\theta^2}{r} \tag{3}$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{4}$$

Integrate both sides of equation (4) partially with respect to z to obtain $p = p(r,z)$.

$$p(r,z) = -\rho g z + f(r)$$

Here $f(r)$ is an arbitrary function. Differentiate both sides with respect to r.

$$\frac{\partial p}{\partial r} = f'(r)$$

Use equation (3) to determine $f(r)$.

$$\begin{aligned}
f'(r) &= \rho\frac{v_\theta^2}{r} \\
&= \frac{\rho}{r}\frac{\kappa^4}{(1-\kappa^2)^2}\frac{(R^2 - r^2)^2}{r^2}\Omega_i^2 \\
&= \rho\frac{\kappa^4\Omega_i^2}{(1-\kappa^2)^2}\frac{R^4 - 2R^2r^2 + r^4}{r^3} \\
&= \rho\left(\frac{\kappa^2\Omega_i}{1-\kappa^2}\right)^2\left(\frac{R^4}{r^3} - \frac{2R^2}{r} + r\right)
\end{aligned}$$

Integrate both sides with respect to r.

$$f(r) = \rho\left(\frac{\kappa^2\Omega_i}{1-\kappa^2}\right)^2\left(-\frac{R^4}{2r^2} - 2R^2\ln r + \frac{r^2}{2}\right) + C$$

As a result, the pressure distribution for the case that the inner cylinder is rotating is

$$p(r,z) = -\rho g z + \rho\left(\frac{\kappa^2\Omega_i}{1-\kappa^2}\right)^2\left(-\frac{R^4}{2r^2} - 2R^2\ln r + \frac{r^2}{2}\right) + C. \tag{5}$$

To determine C, we need to know the pressure somewhere. The pressure everywhere along the liquid-air interface is p_{atm}, and this interface ends at $r = R$ and $z = z_R$ on the outer cylinder.

$$p(R, z_R) = p_{\text{atm}} = -\rho g z_R + \rho\left(\frac{\kappa^2\Omega_i}{1-\kappa^2}\right)^2\left(-\frac{R^2}{2} - 2R^2\ln R + \frac{R^2}{2}\right) + C$$

Subtract the respective sides of this equation from those of equation (5) to eliminate C.

$$\begin{aligned}
p(r,z) - p_{\text{atm}} &= -\rho g(z - z_R) + \rho\left(\frac{\kappa^2\Omega_i}{1-\kappa^2}\right)^2\left[-\frac{R^4}{2r^2} - 2R^2(\ln r - \ln R) + \frac{r^2}{2}\right] \\
&= -\rho g(z - z_R) + \rho\left(\frac{\kappa^2\Omega_i}{1-\kappa^2}\right)^2\left(-\frac{R^4}{2r^2} - 2R^2\ln\frac{r}{R} + \frac{r^2}{2}\right) \\
&= -\rho g(z - z_R) + \rho\left(\frac{\kappa^2\Omega_i}{1-\kappa^2}\right)^2\left(-\frac{R^2}{2}\right)\left(\frac{R^2}{r^2} + 4\ln\frac{r}{R} - \frac{r^2}{R^2}\right) \\
&= -\rho g(z - z_R) - \frac{\rho}{2}\left(\frac{\kappa^2\Omega_i R}{1-\kappa^2}\right)^2\left(\frac{R^2}{r^2} + 4\ln\frac{r}{R} - \frac{r^2}{R^2}\right)
\end{aligned}$$

On the liquid-air interface $p(r,z) = p_{\text{atm}}$. Solve the resulting equation for the shape of the liquid's surface $z = z(r)$.

$$\begin{aligned}
0 &= -\rho g(z - z_R) - \frac{\rho}{2}\left(\frac{\kappa^2\Omega_i R}{1-\kappa^2}\right)^2\left(\frac{R^2}{r^2} + 4\ln\frac{r}{R} - \frac{r^2}{R^2}\right) \\
\rho g(z - z_R) &= -\frac{\rho}{2}\left(\frac{\kappa^2\Omega_i R}{1-\kappa^2}\right)^2\left(\frac{R^2}{r^2} + 4\ln\frac{r}{R} - \frac{r^2}{R^2}\right)
\end{aligned}$$

Divide both sides by $-\rho g$.

$$z_R - z = \frac{1}{2g}\left(\frac{\kappa^2\Omega_i R}{1-\kappa^2}\right)^2\left(\frac{R^2}{r^2} + 4\ln\frac{r}{R} - \frac{r^2}{R^2}\right)$$

Therefore, using ξ for r/R,

$$z_R - z = \frac{1}{2g}\left(\frac{\kappa^2\Omega_i R}{1-\kappa^2}\right)^2(\xi^{-2} + 4\ln\xi - \xi^2).$$

Part (b)

The general solution for the velocity profile was found to be

$$v_\theta(r) = C_1\frac{r}{2} + \frac{C_2}{r}.$$

Apply the two boundary conditions here to determine C_1 and C_2 for the case that only the outer cylinder is rotating.

$$\begin{aligned}
v_\theta(\kappa R) &= C_1\frac{\kappa R}{2} + \frac{C_2}{\kappa R} = 0 \\
v_\theta(R) &= C_1\frac{R}{2} + \frac{C_2}{R} = \Omega_o R
\end{aligned}$$

Solving the system of equations yields

$$C_1 = \frac{2\Omega_o}{1-\kappa^2} \quad \text{and} \quad C_2 = -\frac{\kappa^2}{1-\kappa^2}R^2\Omega_o,$$

which means

$$\begin{aligned}
v_\theta(r) &= \left(\frac{2\Omega_o}{1-\kappa^2}\right)\frac{r}{2} + \left(-\frac{\kappa^2}{1-\kappa^2}R^2\Omega_o\right)\frac{1}{r} \\
&= \frac{\kappa}{1-\kappa^2}\left(\frac{r}{\kappa R} - \frac{\kappa R}{r}\right)\Omega_o R \\
&= \frac{\kappa}{1-\kappa^2}\left(\frac{r^2 - \kappa^2 R^2}{\kappa R r}\right)\Omega_o R \\
&= \frac{1}{1-\kappa^2}\left(\frac{r^2-\kappa^2R^2}{r}\right)\Omega_o.
\end{aligned}$$

Now that we have the velocity profile, we can use the r- and z-equations of motion to determine the pressure distribution.

$$\begin{aligned}
-\rho\frac{v_\theta^2}{r} &= -\frac{\partial p}{\partial r} \\
0 &= -\frac{\partial p}{\partial z} + \rho g_z
\end{aligned}$$

The z-axis points up the cylinders' axis of symmetry, which is opposite the direction of gravity: $\mathbf{g} = -g\hat{\mathbf{z}}$.

$$\frac{\partial p}{\partial r} = \rho\frac{v_\theta^2}{r} \tag{6}$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{7}$$

Integrate both sides of equation (7) partially with respect to z to obtain $p = p(r,z)$.

$$p(r,z) = -\rho g z + g(r)$$

Here $g(r)$ is an arbitrary function. Differentiate both sides with respect to r.

$$\frac{\partial p}{\partial r} = g'(r)$$

Use equation (6) to determine $g(r)$.

$$\begin{aligned}
g'(r) &= \rho\frac{v_\theta^2}{r} \\
&= \frac{\rho}{r}\frac{1}{(1-\kappa^2)^2}\frac{(r^2-\kappa^2R^2)^2}{r^2}\Omega_o^2 \\
&= \rho\frac{\Omega_o^2}{(1-\kappa^2)^2}\frac{r^4 - 2\kappa^2R^2r^2 + \kappa^4R^4}{r^3} \\
&= \rho\frac{\Omega_o^2}{(1-\kappa^2)^2}\left(r - \frac{2\kappa^2R^2}{r} + \frac{\kappa^4R^4}{r^3}\right)
\end{aligned}$$

Integrate both sides with respect to r.

$$g(r) = \rho \frac{\Omega_o^2}{(1-\kappa^2)^2}\left(\frac{r^2}{2} - 2\kappa^2 R^2 \ln r - \frac{\kappa^4 R^4}{2r^2}\right) + D$$

As a result, the pressure distribution for the case that the outer cylinder is rotating is

$$p(r,z) = -\rho g z + \rho \frac{\Omega_o^2}{(1-\kappa^2)^2}\left(\frac{r^2}{2} - 2\kappa^2 R^2 \ln r - \frac{\kappa^4 R^4}{2r^2}\right) + D. \tag{8}$$

To determine D, we need to know the pressure somewhere. The pressure everywhere along the liquid-air interface is p_{atm}, and this interface ends at $r = R$ and $z = z_R$ on the outer cylinder.

$$p(R, z_R) = p_{\text{atm}} = -\rho g z_R + \rho \frac{\Omega_o^2}{(1-\kappa^2)^2}\left(\frac{R^2}{2} - 2\kappa^2 R^2 \ln R - \frac{\kappa^4 R^2}{2}\right) + D$$

Subtract the respective sides of this equation from those of equation (8) to eliminate D.

$$\begin{aligned}
p(r,z) - p_{\text{atm}} &= -\rho g(z - z_R) + \rho \frac{\Omega_o^2}{(1-\kappa^2)^2}\left[\frac{1}{2}(r^2 - R^2) - 2\kappa^2 R^2(\ln r - \ln R) - \frac{\kappa^4 R^2}{2}\left(\frac{R^2}{r^2} - 1\right)\right] \\
&= -\rho g(z - z_R) + \rho \frac{\Omega_o^2}{(1-\kappa^2)^2}\left[\frac{R^2}{2}\left(\frac{r^2}{R^2} - 1\right) - 2\kappa^2 R^2 \ln \frac{r}{R} - \frac{\kappa^4 R^2}{2}\left(\frac{R^2}{r^2} - 1\right)\right] \\
&= -\rho g(z - z_R) + \rho \frac{\Omega_o^2}{(1-\kappa^2)^2}\left(-\frac{\kappa^4 R^2}{2}\right)\left[-\frac{1}{\kappa^4}\left(\frac{r^2}{R^2} - 1\right) + \frac{4}{\kappa^2}\ln\frac{r}{R} + \left(\frac{R^2}{r^2} - 1\right)\right] \\
&= -\rho g(z - z_R) - \frac{\rho}{2}\frac{\kappa^4 \Omega_o^2 R^2}{(1-\kappa^2)^2}\left[-\kappa^{-4}\left(\frac{r^2}{R^2} - 1\right) + 4\kappa^{-2}\ln\frac{r}{R} + \left(\frac{R^2}{r^2} - 1\right)\right] \\
&= -\rho g(z - z_R) - \frac{\rho}{2}\left(\frac{\kappa^2 \Omega_o R}{1-\kappa^2}\right)^2\left[\left(\frac{R^2}{r^2} - 1\right) + 4\kappa^{-2}\ln\frac{r}{R} - \kappa^{-4}\left(\frac{r^2}{R^2} - 1\right)\right]
\end{aligned}$$

On the liquid-air interface $p(r,z) = p_{\text{atm}}$. Solve the resulting equation for the shape of the liquid's surface $z = z(r)$.

$$0 = -\rho g(z - z_R) - \frac{\rho}{2}\left(\frac{\kappa^2 \Omega_o R}{1-\kappa^2}\right)^2\left[\left(\frac{R^2}{r^2} - 1\right) + 4\kappa^{-2}\ln\frac{r}{R} - \kappa^{-4}\left(\frac{r^2}{R^2} - 1\right)\right]$$

$$\rho g(z - z_R) = -\frac{\rho}{2}\left(\frac{\kappa^2 \Omega_o R}{1-\kappa^2}\right)^2\left[\left(\frac{R^2}{r^2} - 1\right) + 4\kappa^{-2}\ln\frac{r}{R} - \kappa^{-4}\left(\frac{r^2}{R^2} - 1\right)\right]$$

Divide both sides by $-\rho g$.

$$z_R - z = \frac{1}{2g}\left(\frac{\kappa^2 \Omega_o R}{1-\kappa^2}\right)^2\left[\left(\frac{R^2}{r^2} - 1\right) + 4\kappa^{-2}\ln\frac{r}{R} - \kappa^{-4}\left(\frac{r^2}{R^2} - 1\right)\right]$$

Therefore, using ξ for r/R,

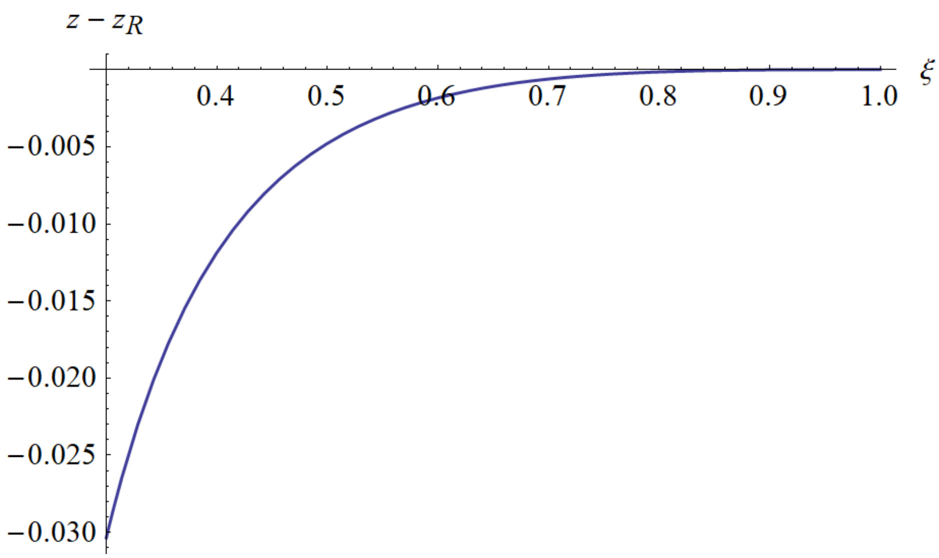
$$z_R - z = \frac{1}{2g}\left(\frac{\kappa^2 \Omega_o R}{1-\kappa^2}\right)^2[(\xi^{-2} - 1) + 4\kappa^{-2}\ln\xi - \kappa^{-4}(\xi^2 - 1)].$$

Part (c)

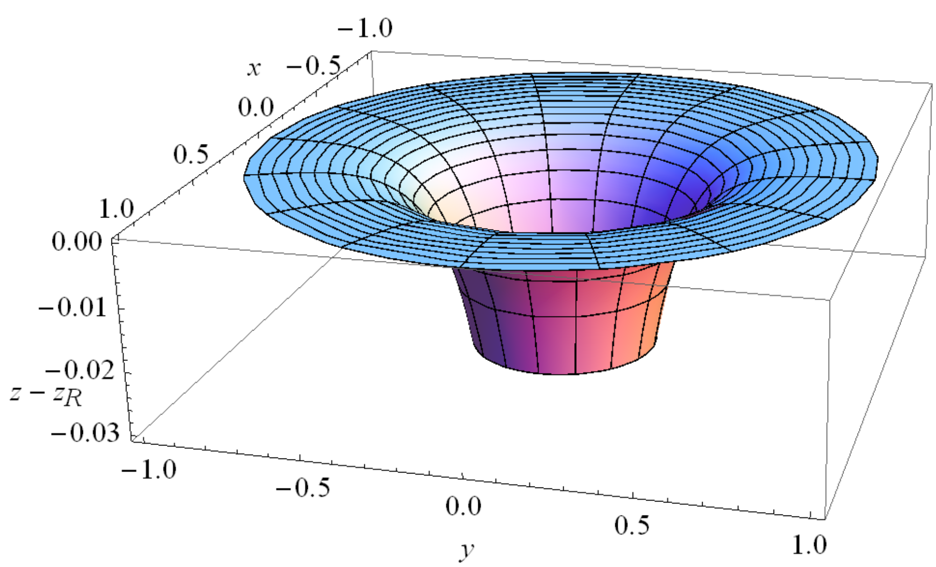
Let $g = 1$, $R = 1$, $\Omega_i = 1$, and $\kappa = 0.3$ in the result of part (a).

$$z - z_R = -\frac{1}{2}\left(\frac{0.3^2}{1 - 0.3^2}\right)^2 (\xi^{-2} + 4\ln\xi - \xi^2)$$

A plot of $z - z_R$ versus ξ is shown below.



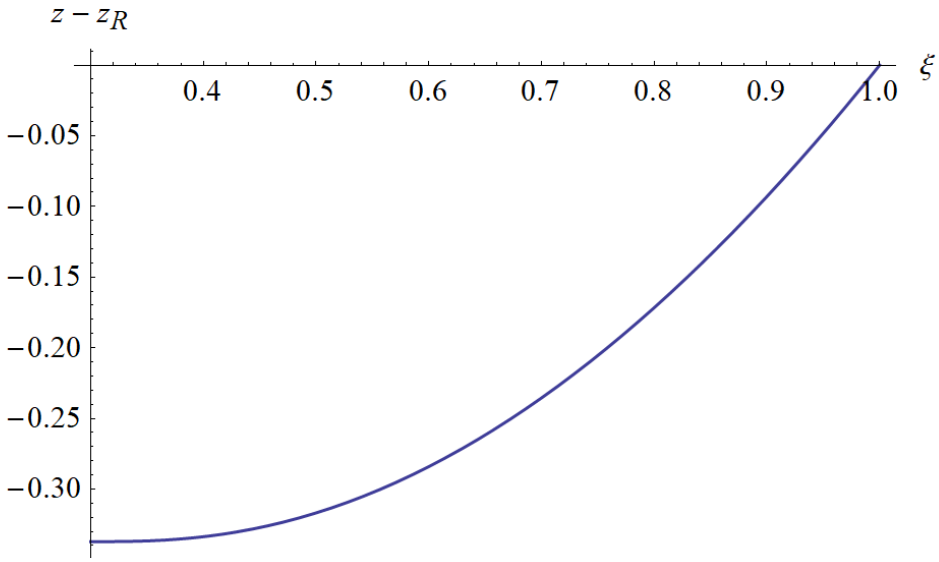
The numbers on the axes are not important. What is important is the shape of the curve. Rotate it about the z-axis to get a three-dimensional visual of the liquid-air interface for the case that the inner cylinder is rotating and the outer cylinder is stationary.



Let $g = 1$, $R = 1$, $\Omega_o = 1$, and $\kappa = 0.3$ in the result of part (b).

$$z - z_R = -\frac{1}{2}\left(\frac{0.3^2}{1-0.3^2}\right)^2 [(\xi^{-2} - 1) + 4 \times 0.3^{-2} \ln \xi - 0.3^{-4}(\xi^2 - 1)]$$

A plot of $z - z_R$ versus ξ is shown below.



The numbers on the axes are not important. What is important is the shape of the curve. Rotate it about the z-axis to get a three-dimensional visual of the liquid-air interface for the case that the outer cylinder is rotating and the inner cylinder is stationary.