Problem 3B.8

Velocity distribution for creeping flow toward a slot (Fig. 3B.7).[4] It is desired to get the velocity distribution given for the upstream region in the previous problem. We postulate that $v_\theta = 0$, $v_z = 0$, $v_r = v_r(r, \theta)$, and $\mathscr{P} = \mathscr{P}(r, \theta)$.

(a) Show that the equation of continuity in cylindrical coordinates gives $v_r = f(\theta)/r$, where $f(\theta)$ is a function of θ for which $df/d\theta = 0$ at $\theta = 0$, and $f = 0$ at $\theta = \pi/2$.

(b) Write the r- and θ-components of the creeping flow equation of motion, and insert the expression for $f(\theta)$ from (a).

(c) Differentiate the r-component of the equation of motion with respect to θ and the θ-component with respect to r. Show that this leads to

$$\frac{d^3 f}{d\theta^3} + 4\frac{df}{d\theta} = 0 \tag{3B.8-1}$$

(d) Solve this differential equation and obtain an expression for $f(\theta)$ containing three integration constants.

(e) Evaluate the integration constants by using the two boundary conditions in (a) and the fact that the total mass-flow rate through any cylindrical surface must equal w. This gives

$$v_r = -\frac{2w}{\pi W \rho r}\cos^2\theta \tag{3B.8-2}$$

(f) Next from the equations of motion in (b) obtain $\mathscr{P}(r, \theta)$ as

$$\mathscr{P}(r, \theta) = \mathscr{P}_\infty - \frac{2\mu w}{\pi W \rho r^2}\cos 2\theta \tag{3B.8-3}$$

What is the physical meaning of $\mathscr{P}_\infty$?

(g) Show that the total normal stress exerted on the solid surface at $\theta = \pi/2$ is

$$(p + \tau_{\theta\theta})|_{\theta=\pi/2} = p_\infty + \frac{2\mu w}{\pi W \rho r^2} \tag{3B.8-4}$$

(h) Next evaluate $\tau_{\theta r}$ on the same solid surface.

(i) Show that the velocity profile obtained in Eq. 3B.8-2 is the equivalent to Eqs. 3B.7-2 and 3.

Solution

[4] Adapted from R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1, Wiley-Interscience, New York, 2nd edition (1987), pp. 42–43.

Part (a)

Fig. 3B.7 is shown below, which illustrates the flow and the chosen coordinate system.

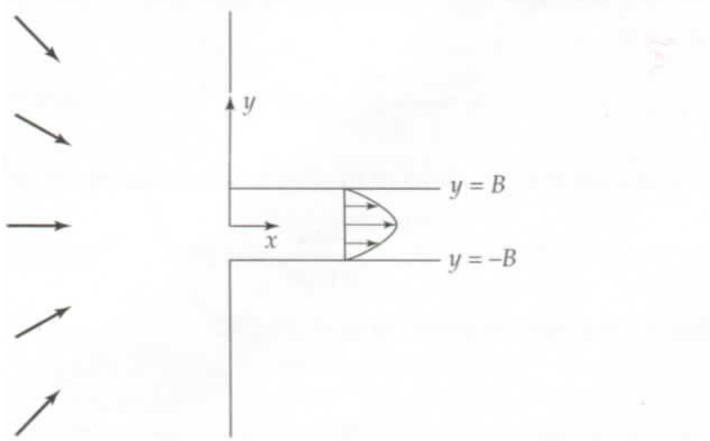


Fig. 3B.7. Flow of a liquid into a slot from a semi-infinite region $x < 0$.

The velocity is assumed to only have a radial component that varies with r and θ.

$$\mathbf{v} = v_r(r, \theta)\hat{\mathbf{r}}$$

In addition, the pressure is assumed to vary at each point in the half-plane.

$$p = p(r, \theta)$$

Two boundary conditions can be obtained from the assumption that the fluid does not slip on the walls. Since we only care about the flow outside the slot, the boundary conditions associated with the walls at $y = \pm B$ do not count.

$$\text{Boundary Condition 1:} \quad v_r\left(r, \frac{\pi}{2}\right) = 0$$

$$\text{Boundary Condition 2:} \quad v_r\left(r, \frac{3\pi}{2}\right) = 0$$

Another boundary condition is obtained from the fact that the flow is symmetric about the line which is collinear with the x-axis.

$$\text{Boundary Condition 3:} \quad \frac{\partial v_r}{\partial \theta}(r, \pi) = 0$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

Expand the left side in cylindrical coordinates, using the formula in Appendix B.4 on page 846.

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=\,0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=\,0} = 0$$

Multiply both sides by r.

$$\frac{\partial}{\partial r}(rv_r) = 0$$

Integrate both sides partially with respect to r.

$$rv_r = f(\theta)$$

Therefore, dividing both sides by r,

$$v_r = \frac{f(\theta)}{r},$$

where f is an arbitrary differentiable function of θ. Apply the three boundary conditions here to get three boundary conditions for f.

$$\begin{aligned}
v_r\left(r, \frac{\pi}{2}\right) = 0 \quad &\rightarrow \quad \frac{f\left(\frac{\pi}{2}\right)}{r} = 0 \quad \rightarrow \quad f\left(\frac{\pi}{2}\right) = 0 \\
v_r\left(r, \frac{3\pi}{2}\right) = 0 \quad &\rightarrow \quad \frac{f\left(\frac{3\pi}{2}\right)}{r} = 0 \quad \rightarrow \quad f\left(\frac{3\pi}{2}\right) = 0 \\
\frac{\partial v_r}{\partial \theta}(r, \pi) = 0 \quad &\rightarrow \quad \frac{f'(\pi)}{r} = 0 \quad \rightarrow \quad f'(\pi) = 0
\end{aligned}$$

Part (b)

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

As this is a vector equation, it actually represents three scalar equations, one for each variable in the chosen coordinate system. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\bigg(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{v_\theta^2}{r}}_{=0}\bigg) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\right] + \rho g_r$$

$$\rho\bigg(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\bigg) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta$$

$$\rho\bigg(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\bigg) = \underbrace{-\frac{\partial p}{\partial z}}_{=0} + \mu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \underbrace{\rho g_z}_{=0}$$

Note that $v_r(\partial v_r/\partial r) = 0$ because of the creeping flow assumption—all acceleration terms are neglected. Substitute $v_r = f(\theta)/r$ into the relevant equations.

$$\begin{aligned}
0 &= -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}[f(\theta)]\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{f(\theta)}{r}\right)\right] + \rho g_r \\
0 &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{2\mu}{r^2}\frac{\partial}{\partial \theta}\left(\frac{f(\theta)}{r}\right) + \rho g_\theta
\end{aligned}$$

Evaluate the derivatives.

$$0 = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r^3}f''(\theta)\right] + \rho g_r$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{2\mu}{r^3}f'(\theta) + \rho g_\theta$$

Multiply both sides of the second equation by $-r$.

$$0 = -\frac{\partial p}{\partial r} + \frac{\mu}{r^3}f''(\theta) + \rho g_r \tag{1}$$

$$0 = \frac{\partial p}{\partial \theta} - \frac{2\mu}{r^2}f'(\theta) - \rho g_\theta r \tag{2}$$

In this problem gravity points straight down: $\mathbf{g} = -g\hat{\mathbf{y}}$. Write this unit vector in terms of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ by using formula A.6-14 on page 827.

$$\mathbf{g} = -g[(\sin\theta)\hat{\mathbf{r}} + (\cos\theta)\hat{\boldsymbol{\theta}}] = -g(\sin\theta)\hat{\mathbf{r}} - g(\cos\theta)\hat{\boldsymbol{\theta}}$$

We see that $g_r = -g\sin\theta$ and $g_\theta = -g\cos\theta$. Plug these results into equations (1) and (2).

$$0 = -\frac{\partial p}{\partial r} + \frac{\mu}{r^3}f''(\theta) - \rho g\sin\theta$$

$$0 = \frac{\partial p}{\partial \theta} - \frac{2\mu}{r^2}f'(\theta) + \rho g r\cos\theta$$

Factor the minus sign in the first equation and rearrange the terms in the second equation.

$$0 = -\left(\frac{\partial p}{\partial r} + \rho g\sin\theta\right) + \frac{\mu}{r^3}f''(\theta)$$

$$0 = \frac{\partial p}{\partial \theta} + \rho g r\cos\theta - \frac{2\mu}{r^2}f'(\theta)$$

Factor the derivative operator in each equation.

$$0 = -\frac{\partial}{\partial r}(p + \rho g r\sin\theta) + \frac{\mu}{r^3}f''(\theta)$$

$$0 = \frac{\partial}{\partial \theta}(p + \rho g r\sin\theta) - \frac{2\mu}{r^2}f'(\theta)$$

Now introduce the modified pressure function $\mathscr{P}(r,\theta) = p(r,\theta) + \rho g r\sin\theta$. For a chosen point (r,θ) in the half-plane, the modified pressure is the corresponding pressure at the projection onto the x-axis.

$$0 = -\frac{\partial \mathscr{P}}{\partial r} + \frac{\mu}{r^3}f''(\theta) \tag{3}$$

$$0 = \frac{\partial \mathscr{P}}{\partial \theta} - \frac{2\mu}{r^2}f'(\theta) \tag{4}$$

Part (c)

Differentiate both sides of equation (3) with respect to θ, and differentiate both sides of equation (4) with respect to r.

$$0 = -\frac{\partial^2 \mathscr{P}}{\partial \theta \partial r} + \frac{\mu}{r^3}f'''(\theta)$$

$$0 = \frac{\partial^2 \mathscr{P}}{\partial r \partial \theta} + \frac{4\mu}{r^3}f'(\theta)$$

The mixed partial derivatives are equal by Clairaut's theorem, so they cancel when the respective sides of each equation are added together.

$$0 = \frac{\mu}{r^3} f'''(\theta) + \frac{4\mu}{r^3} f'(\theta)$$

Therefore, multiplying both sides by r^3/μ,

$$f''' + 4f' = 0.$$

Part (d)

This is a linear homogeneous ODE which has constant coefficients, so its solutions are of the form $f = e^{k\theta}$.

$$f = e^{k\theta} \quad \rightarrow \quad f' = ke^{k\theta} \quad \rightarrow \quad f'' = k^2 e^{k\theta} \quad \rightarrow \quad f''' = k^3 e^{k\theta}$$

Substitute these formulas into the ODE to determine k.

$$k^3 e^{k\theta} + 4(ke^{k\theta}) = 0$$

Divide both sides by $e^{k\theta}$.

$$\begin{gathered} k^3 + 4k = 0 \\ k(k^2 + 4) = 0 \\ k = \{0, -2i, 2i\} \end{gathered}$$

Three solutions to the ODE are then $f = e^0 = 1$ and $f = e^{-2i\theta}$ and $f = e^{2i\theta}$. According to the principle of superposition, the general solution is a linear combination of these three.

$$\begin{aligned} f(\theta) &= C_1 + C_2 e^{-2i\theta} + C_3 e^{2i\theta} \\ &= C_1 + C_2[\cos(-2\theta) + i\sin(-2\theta)] + C_3[\cos(2\theta) + i\sin(2\theta)] \\ &= C_1 + C_2[\cos(2\theta) - i\sin(2\theta)] + C_3[\cos(2\theta) + i\sin(2\theta)] \\ &= C_1 + (C_2 + C_3)\cos 2\theta + (-iC_2 + iC_3)\sin 2\theta \end{aligned}$$

Therefore, using new arbitrary constants,

$$f(\theta) = C_1 + C_4 \cos 2\theta + C_5 \sin 2\theta.$$

Part (e)

Take a derivative of the general solution.

$$f'(\theta) = -2C_4 \sin 2\theta + 2C_5 \cos 2\theta$$

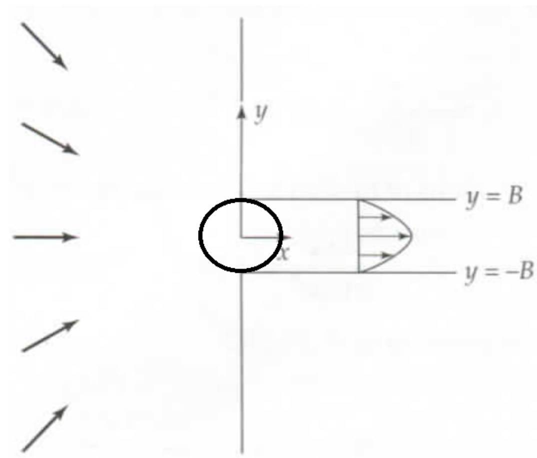
Now apply the three boundary conditions for f to determine C_1, C_4, and C_5.

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= C_1 + C_4 \cos \pi + C_5 \sin \pi = C_1 - C_4 = 0 \\ f\left(\frac{3\pi}{2}\right) &= C_1 + C_4 \cos 3\pi + C_5 \sin 3\pi = C_1 - C_4 = 0 \\ f'(\pi) &= -2C_4 \sin 2\pi + 2C_5 \cos 2\pi = 2C_5 = 0 \end{aligned}$$

Solving this system of equations yields $C_5 = 0$ and $C_4 = C_1$.

$$\begin{aligned} f(\theta) &= C_1 + C_1 \cos 2\theta \\ &= C_1(1 + \cos 2\theta) \end{aligned}$$

Apparently, the $f(3\pi/2) = 0$ boundary condition is redundant. But that's okay because we can use the fact that the mass flow rate through any cylindrical surface is w to get another one. Consider a cylindrical shell centered at the origin with radius B and width W. Its axis of symmetry lies along the z-axis (through the paper).



Observe that all the fluid must pass through this shell to enter the slot. Start with the formula for the volumetric flow rate.

$$\frac{dV}{dt} = \mathbf{v} \cdot \mathbf{A}$$

Since the velocity varies over the shell's surface, an integral is necessary.

$$\frac{dV}{dt} = \int \mathbf{v} \cdot d\mathbf{A}$$

The outward unit vector perpendicular to the shell is $\hat{\mathbf{r}}$, so the dot product yields $-v_r\,dA$. The minus sign is included because the fluid is moving radially inward, antiparallel to the area vector.

$$\frac{dV}{dt} = \int (-v_r)\,dA$$

Multiply both sides by ρ.

$$\rho \frac{dV}{dt} = \rho \int (-v_r)\,dA$$

Bring ρ inside the derivative.

$$\frac{d(\rho V)}{dt} = -\rho \int v_r\,dA$$

Density times volume is mass.

$$\frac{dm}{dt} = -\rho \int v_r\,dA$$

The mass flow rate is w.

$$w = -\rho \int v_r\,dA$$

For this shell, the area differential is a bit of arclength ds times the width W, that is, $dA = W\,ds = WB\,d\theta$. Also, on the shell $v_r = v_r(B,\theta)$.

$$\begin{aligned} w &= -\rho \int_{\pi/2}^{3\pi/2} v_r(B,\theta)(WB\,d\theta) \\ &= -\rho \int_{\pi/2}^{3\pi/2} \frac{f(\theta)}{B}(WB\,d\theta) \\ &= -\rho W \int_{\pi/2}^{3\pi/2} f(\theta)\,d\theta \\ &= -\rho W \int_{\pi/2}^{3\pi/2} C_1(1+\cos 2\theta)\,d\theta \\ &= -\rho W C_1 \left(\int_{\pi/2}^{3\pi/2} d\theta + \int_{\pi/2}^{3\pi/2} \cos 2\theta\,d\theta \right) \\ &= -\rho W C_1 (\pi + 0) \\ &= -\rho W C_1 \pi \end{aligned}$$

Solve for C_1.

$$C_1 = -\frac{w}{\rho W \pi}$$

This means that

$$\begin{aligned} f(\theta) &= C_1(1+\cos 2\theta) \\ &= C_1(2\cos^2\theta) \\ &= -\frac{w}{\rho W \pi}(2\cos^2\theta) \\ &= -\frac{2w}{\pi W \rho}\cos^2\theta. \end{aligned}$$

Therefore, since $v_r = f(\theta)/r$,

$$v_r = -\frac{2w}{\pi W \rho r}\cos^2\theta.$$

Part (f)

Differentiate $f(\theta)$ twice to get f' and f''.

$$\begin{aligned} f'(\theta) &= -\frac{2w}{\pi W \rho}[2\cos\theta(-\sin\theta)] \\ &= \frac{2w}{\pi W \rho}\sin 2\theta \\ f''(\theta) &= \frac{4w}{\pi W \rho}\cos 2\theta \end{aligned}$$

Substitute these formulas into equations (3) and (4), the results of part (b).

$$0 = -\frac{\partial \mathscr{P}}{\partial r} + \frac{\mu}{r^3}f''(\theta) \quad \rightarrow \quad \frac{\partial \mathscr{P}}{\partial r} = \frac{\mu}{r^3}f''(\theta) \quad \rightarrow \quad \frac{\partial \mathscr{P}}{\partial r} = \frac{4\mu w}{\pi W \rho r^3}\cos 2\theta \tag{5}$$

$$0 = \frac{\partial \mathscr{P}}{\partial \theta} - \frac{2\mu}{r^2}f'(\theta) \quad \rightarrow \quad \frac{\partial \mathscr{P}}{\partial \theta} = \frac{2\mu}{r^2}f'(\theta) \quad \rightarrow \quad \frac{\partial \mathscr{P}}{\partial \theta} = \frac{4\mu w}{\pi W \rho r^2}\sin 2\theta \tag{6}$$

Integrate both sides of equation (6) partially with respect to θ to get $\mathscr{P}$.

$$\mathscr{P}(r,\theta) = \frac{4\mu w}{\pi W \rho r^2}\left(-\frac{1}{2}\cos 2\theta\right) + h(r)$$
$$= -\frac{2\mu w}{\pi W \rho r^2}\cos 2\theta + h(r)$$

Differentiate both sides with respect to r.

$$\frac{\partial \mathscr{P}}{\partial r} = \frac{4\mu w}{\pi W \rho r^3}\cos 2\theta + h'(r)$$

Comparing this with equation (5), we see that

$$h'(r) = 0.$$

Integrate both sides with respect to r.

$$h(r) = \mathscr{P}_\infty$$

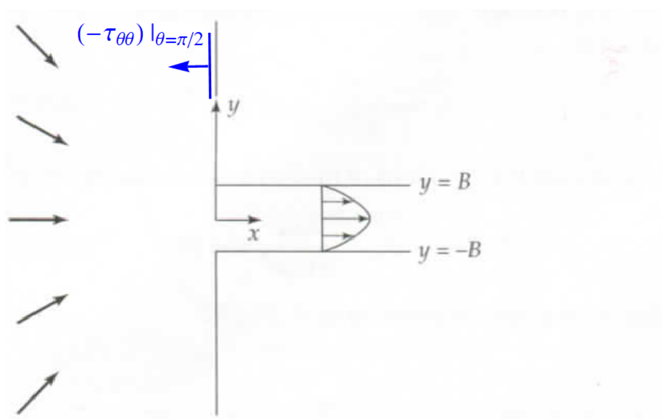
Therefore,

$$\mathscr{P}(r,\theta) = \mathscr{P}_\infty - \frac{2\mu w}{\pi W \rho r^2}\cos 2\theta,$$

where $\mathscr{P}_\infty$ is the pressure at the projection onto the x-axis of a point far, far away from the slot.

Part (g)

Here we want to calculate the total normal stress on the vertical wall directly above the slot. $-\tau_{\theta\theta}$ represents the viscous force in the θ-direction on a unit area perpendicular to the θ-direction. The minus sign in front accounts for the fact that the fluid is in a region of greater θ acting on a wall of lesser θ. No second minus sign is needed to account for the direction that the fluid is flowing in because the velocity components that $\tau_{\theta\theta}$ is in terms of already take care of it.



In addition to viscous stress, the pressure also contributes to the normal stress on the wall. The total is the sum of these two.

$$\text{Total Normal Stress on Wall} = [-p + (-\tau_{\theta\theta})]|_{\theta=\pi/2} = -(p + \tau_{\theta\theta})|_{\theta=\pi/2}$$

There's a minus sign in front of p because the force on the wall that results from the pressure acts in the negative θ-direction.

Since we only care about the magnitude of the stress on the wall, the minus sign will be dropped.

$$\begin{aligned}
(p+\tau_{\theta\theta})|_{\theta=\pi/2} &= (\mathscr{P} - \rho g r\sin\theta + \tau_{\theta\theta})|_{\theta=\pi/2} \\
&= \left.\left(\mathscr{P}_\infty - \frac{2\mu w}{\pi W\rho r^2}\cos 2\theta - \rho g r\sin\theta + \tau_{\theta\theta}\right)\right|_{\theta=\pi/2} \\
&= \mathscr{P}_\infty - \frac{2\mu w}{\pi W\rho r^2}\cos\pi - \rho g r\sin\frac{\pi}{2} + \tau_{\theta\theta}|_{\theta=\pi/2} \\
&= \mathscr{P}_\infty + \frac{2\mu w}{\pi W\rho r^2} - \rho g r + \tau_{\theta\theta}|_{\theta=\pi/2}
\end{aligned}$$

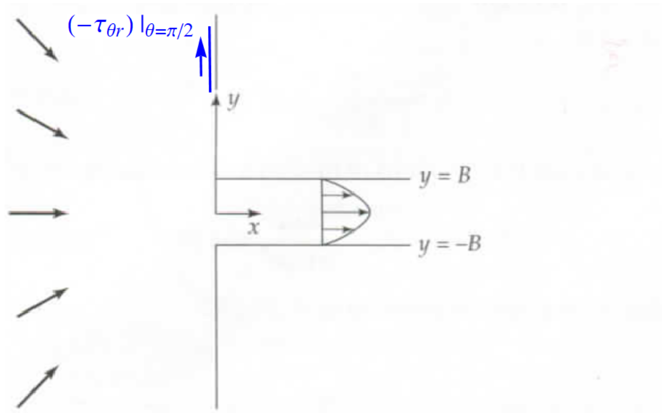
Use the formula for $\tau_{\theta\theta}$ in Appendix B.1 on page 844.

$$\begin{aligned}
(p+\tau_{\theta\theta})|_{\theta=\pi/2} &= \mathscr{P}_\infty + \frac{2\mu w}{\pi W\rho r^2} - \rho g r + \left.\left\{-\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right)\right] + \left(\frac{2}{3}\mu - \kappa\right)(\nabla\cdot\mathbf{v})\right\}\right|_{\theta=\pi/2} \\
&= \mathscr{P}_\infty + \frac{2\mu w}{\pi W\rho r^2} - \rho g r + \left.\left\{-\mu\left[2\left(\frac{1}{r}(0) + \frac{v_r}{r}\right)\right] + \left(\frac{2}{3}\mu - \kappa\right)(0)\right\}\right|_{\theta=\pi/2} \\
&= \mathscr{P}_\infty + \frac{2\mu w}{\pi W\rho r^2} - \rho g r - 2\mu\left.\left(-\frac{2w}{\pi W\rho r^2}\cos^2\theta\right)\right|_{\theta=\pi/2} \\
&= \mathscr{P}_\infty + \frac{2\mu w}{\pi W\rho r^2} - \rho g r + \frac{4\mu w}{\pi W\rho r^2}\cos^2\frac{\pi}{2} \\
&= \mathscr{P}_\infty + \frac{2\mu w}{\pi W\rho r^2} - \rho g r \\
&= (\mathscr{P}_\infty - \rho g r) + \frac{2\mu w}{\pi W\rho r^2} \\
&= p_\infty + \frac{2\mu w}{\pi W\rho r^2}
\end{aligned}$$

Note that $p_\infty = p_\infty(r) = \mathscr{P}_\infty - \rho g r$ is the pressure far to the left at the height r of interest. If we wanted to calculate the normal force on the wall, we would integrate $(p+\tau_{\theta\theta})|_{\theta=\pi/2}$ over the wall's surface area.

Part (h)

Here we want to calculate the vertical shear stress due to viscosity on the wall directly above the slot. $-\tau_{\theta r}$ represents the viscous force in the r-direction on a unit area perpendicular to the θ-direction. As with $\tau_{\theta\theta}$, a minus sign is in front of $\tau_{\theta r}$ to account for the fact that the fluid is in a region of greater θ acting on a wall of lesser θ.



Use the formula for $\tau_{\theta r}$ in Appendix B.1 on page 844.

$$\begin{aligned}
(-\tau_{\theta r})|_{\theta=\pi/2} &= \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\bigg|_{\theta=\pi/2} \\
&= \mu\left[r\frac{\partial}{\partial r}(0)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(-\frac{2w}{\pi W\rho r}\cos^2\theta\right)\right]\bigg|_{\theta=\pi/2} \\
&= \mu\left[0+\frac{1}{r}\left(\frac{4w}{\pi W\rho r}\cos\theta\sin\theta\right)\right]\bigg|_{\theta=\pi/2} \\
&= \mu\left[0+\frac{1}{r}(0)\right]\bigg|_{\theta=\pi/2} \\
&= 0
\end{aligned}$$

Therefore, the vertical shear stress on the wall is zero.

Part (i)

The velocity was found to be

$$\mathbf{v} = v_r(r,\theta)\hat{\mathbf{r}} = -\frac{2w}{\pi W\rho r}(\cos^2\theta)\hat{\mathbf{r}}.$$

Write the radial unit vector $\hat{\mathbf{r}}$ in terms of the Cartesian unit vectors, $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, by using formula A.6-10 on page 827.

$$\begin{aligned}\mathbf{v} &= -\frac{2w}{\pi W\rho r}(\cos^2\theta)[(\cos\theta)\hat{\mathbf{x}} + (\sin\theta)\hat{\mathbf{y}}] \\ &= -\frac{2w}{\pi W\rho r}(\cos^3\theta)\hat{\mathbf{x}} - \frac{2w}{\pi W\rho r}(\cos^2\theta)(\sin\theta)\hat{\mathbf{y}}\end{aligned}$$

Use the facts that $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}\mathbf{v} &= -\frac{2w}{\pi W\rho r}\left(\frac{x^3}{r^3}\right)\hat{\mathbf{x}} - \frac{2w}{\pi W\rho r}\left(\frac{x^2}{r^2}\right)\left(\frac{y}{r}\right)\hat{\mathbf{y}} \\ &= -\frac{2w}{\pi W\rho}\left(\frac{x^3}{r^4}\right)\hat{\mathbf{x}} - \frac{2w}{\pi W\rho}\left(\frac{x^2 y}{r^4}\right)\hat{\mathbf{y}}\end{aligned}$$

Use the fact that $r^2 = x^2 + y^2$.

$$\mathbf{v} = -\frac{2w}{\pi W\rho}\frac{x^3}{(x^2+y^2)^2}\hat{\mathbf{x}} - \frac{2w}{\pi W\rho}\frac{x^2 y}{(x^2+y^2)^2}\hat{\mathbf{y}}$$

Therefore, the components of velocity in Cartesian coordinates outside the slot for creeping flow are

$$\begin{aligned}v_x &= -\frac{2w}{\pi W\rho}\frac{x^3}{(x^2+y^2)^2} \\ v_y &= -\frac{2w}{\pi W\rho}\frac{x^2 y}{(x^2+y^2)^2} \\ v_z &= 0.\end{aligned}$$