Problem 3D.1

Derivation of the equations of change by integral theorems (Fig. 3D.1).

(a) A fluid is flowing through some region of 3-dimensional space. Select an arbitrary "blob" of this fluid—that is, a region that is bounded by some surface $S(t)$ enclosing a volume $V(t)$, whose elements move with the local fluid velocity.

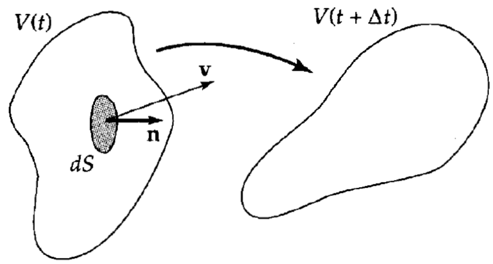


Fig. 3D.1. Moving "blob" of fluid to which Newton's second law of motion is applied. Every element of the fluid surface $dS(t)$ of the moving, deforming volume element $V(t)$ moves with the local, instantaneous fluid velocity $\mathbf{v}(t)$.

Apply Newton's second law of motion to this system to get

$$\frac{d}{dt}\int_{V(t)} \rho\mathbf{v}\,dV = -\int_{S(t)} [\mathbf{n}\cdot\boldsymbol{\pi}]\,dS + \int_{V(t)} \rho\mathbf{g}\,dV \tag{3D.1-1}$$

in which the terms on the right account for the surface and volume forces acting on the system. Apply the Leibniz formula for differentiating an integral (see §A.5), recognizing that at all points on the surface of the blob, the surface velocity is identical to the fluid velocity. Next apply the Gauss theorem for a tensor (see §A.5) so that each term in the equation is a volume integral. Since the choice of the "blob" is arbitrary, all the integral signs may be removed, and the equation of motion in Eq. 3.2-9 is obtained.

(b) Derive the equation of motion by writing a momentum balance over an arbitrary region of volume V and surface S, fixed in space, through which a fluid is flowing. In doing this, just parallel the derivation given in §3.2 for a rectangular fluid element. The Gauss theorem for a tensor is needed to complete the derivation.

This problem shows that applying Newton's second law of motion to an arbitrary moving "blob" of fluid is equivalent to setting up a momentum balance over an arbitrary fixed region of space through which the fluid is moving. Both (a) and (b) give the same result as that obtained in §3.2.

(c) Derive the equation of continuity using a volume element of arbitrary shape, both moving and fixed, by the methods outlined in (a) and (b).

Solution

Back in §3.1 and §3.2, the equation of continuity and the equation of motion, respectively, were derived using a differential formulation. An integral formulation will be used here to derive these equations.

Part (a)

Apply Newton's second law in terms of momentum to the blob on the left in Fig. 3D.1.

$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt}$$

The forces acting on the blob consist of those acting on its body and those acting on its surface.

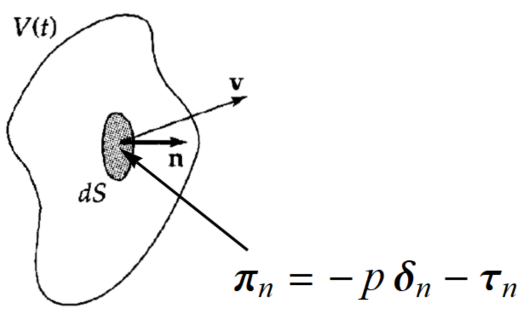
$$\text{Body forces} + \text{Surface forces} = \frac{d\mathbf{p}}{dt}$$

The gravitational force is the one force acting on the body, and the total is obtained by summing (integrating) the gravitational force on each particle in the blob.

$$\int_{\text{blob mass}} \mathbf{g}\,dm + \text{Surface forces} = \frac{d\mathbf{p}}{dt}$$

In general, the second-order tensor $\boldsymbol{\pi} = p\boldsymbol{\delta} + \boldsymbol{\tau}$ contains all the surface forces per unit area of interest, namely the pressure, the normal stress due to viscosity, and the shear stress due to viscosity. To obtain the total force, one needs to integrate $\boldsymbol{\pi}$ over the surface area $S(t)$.

$$\int_{\text{blob mass}} \mathbf{g}\,dm + \int_{S(t)} \boldsymbol{\pi} \cdot d\mathbf{S} = \frac{d\mathbf{p}}{dt}$$



For the blob here, the collection of surface forces on the area element dS acts inward, whereas the normal unit vector $\mathbf{n}$ points outward. A minus sign is needed to account for this disparity.

$$\int_{\text{blob mass}} \mathbf{g}\,dm + \int_{S(t)} (-\boldsymbol{\pi}) \cdot (\mathbf{n}\,dS) = \frac{d\mathbf{p}}{dt}$$

$\mathbf{p}$ is the linear momentum, and the total is obtained by summing (integrating) all the momenta in the blob.

$$\int_{\text{blob mass}} \mathbf{g}\,dm - \int_{S(t)} \boldsymbol{\pi} \cdot \mathbf{n}\,dS = \frac{d}{dt} \int_{\text{blob mass}} \mathbf{v}\,dm$$

Turn these mass integrals into volume integrals by introducing the mass density ρ: $dm = \rho\, dV$.

$$\int_{V(t)} \rho\mathbf{g}\, dV - \int_{S(t)} \boldsymbol{\pi}\cdot\mathbf{n}\, dS = \frac{d}{dt}\int_{V(t)} \rho\mathbf{v}\, dV$$

Apply the three-dimensional analog of the Leibnitz rule (formula A.5-5 on page 824) to differentiate the integral.

$$\int_{V(t)} \rho\mathbf{g}\, dV - \int_{S(t)} \boldsymbol{\pi}\cdot\mathbf{n}\, dS = \int_{V(t)} \frac{\partial}{\partial t}\rho\mathbf{v}\, dV + \int_{S(t)} \rho\mathbf{v}(\mathbf{v}_S\cdot\mathbf{n})\, dS$$

It's assumed that the blob surface $S(t)$ moves with the local fluid velocity so that $\mathbf{v}_S = \mathbf{v}$.

$$\int_{V(t)} \rho\mathbf{g}\, dV - \int_{S(t)} \boldsymbol{\pi}\cdot\mathbf{n}\, dS = \int_{V(t)} \frac{\partial}{\partial t}\rho\mathbf{v}\, dV + \int_{S(t)} \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})\, dS$$

The dot product is associative.

$$\int_{V(t)} \rho\mathbf{g}\, dV - \int_{S(t)} \boldsymbol{\pi}\cdot\mathbf{n}\, dS = \int_{V(t)} \frac{\partial}{\partial t}\rho\mathbf{v}\, dV + \int_{S(t)} \rho\mathbf{v}\mathbf{v}\cdot\mathbf{n}\, dS$$

Apply Gauss's theorem to turn these surface integrals into volume integrals.

$$\int_{V(t)} \rho\mathbf{g}\, dV - \int_{V(t)} \nabla\cdot\boldsymbol{\pi}\, dV = \int_{V(t)} \frac{\partial}{\partial t}\rho\mathbf{v}\, dV + \int_{V(t)} \nabla\cdot\rho\mathbf{v}\mathbf{v}\, dV$$

Since the choice of the blob is arbitrary, all the integrals may be removed.

$$\rho\mathbf{g} - \nabla\cdot\boldsymbol{\pi} = \frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v}$$

Solve for $\partial/\partial t(\rho\mathbf{v})$.

$$\begin{aligned}
\frac{\partial}{\partial t}\rho\mathbf{v} &= -\nabla\cdot\rho\mathbf{v}\mathbf{v} - \nabla\cdot\boldsymbol{\pi} + \rho\mathbf{g} \\
&= -\nabla\cdot\rho\mathbf{v}\mathbf{v} - \nabla\cdot(p\boldsymbol{\delta} + \boldsymbol{\tau}) + \rho\mathbf{g} \\
&= -\nabla\cdot\rho\mathbf{v}\mathbf{v} - \nabla\cdot(p\boldsymbol{\delta}) - \nabla\cdot\boldsymbol{\tau} + \rho\mathbf{g}
\end{aligned}$$

Simplify the second term.

$$\begin{aligned}
\nabla\cdot(p\boldsymbol{\delta}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(p\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\boldsymbol{\delta}_k\delta_{jk}\right) \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(p\sum_{j=1}^{3}\boldsymbol{\delta}_j\boldsymbol{\delta}_j\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)\boldsymbol{\delta}_j\frac{\partial p}{\partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}\boldsymbol{\delta}_j\frac{\partial p}{\partial x_i} \\
&= \sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial p}{\partial x_i} \\
&= \nabla p
\end{aligned}$$

Therefore, Eq. 3.2-9 in the textbook is obtained.

$$\frac{\partial}{\partial t}\rho\mathbf{v} = -\nabla\cdot\rho\mathbf{v}\mathbf{v} - \nabla p - \nabla\cdot\boldsymbol{\tau} + \rho\mathbf{g} \tag{3.2-9}$$

Now use Eq. 1.2-7, Newton's generalized law of viscosity, on page 19 for the viscous stress $\boldsymbol{\tau}$.

$$\boldsymbol{\tau} = -\mu\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger\right) + \left(\frac{2}{3}\mu - \kappa\right)(\nabla\cdot\mathbf{v})\boldsymbol{\delta} \tag{1.2-7}$$

Substitute it into the equation of motion and simplify, making the assumption that μ is constant.

$$\begin{aligned}
\frac{\partial}{\partial t}\rho\mathbf{v} &= -\nabla\cdot\rho\mathbf{v}\mathbf{v} - \nabla p - \nabla\cdot\left\{-\mu\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger\right) + \left(\frac{2}{3}\mu - \kappa\right)(\nabla\cdot\mathbf{v})\boldsymbol{\delta}\right\} + \rho\mathbf{g} \\
&= -\nabla\cdot\rho\mathbf{v}\mathbf{v} - \nabla p + \left\{\mu\nabla\cdot\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger\right) - \left(\frac{2}{3}\mu - \kappa\right)\nabla\cdot[(\nabla\cdot\mathbf{v})\boldsymbol{\delta}]\right\} + \rho\mathbf{g} \\
&= -\nabla\cdot\rho\mathbf{v}\mathbf{v} - \nabla p + \mu\nabla\cdot\nabla\mathbf{v} + \mu\nabla\cdot(\nabla\mathbf{v})^\dagger - \left(\frac{2}{3}\mu - \kappa\right)\nabla\cdot[(\nabla\cdot\mathbf{v})\boldsymbol{\delta}] + \rho\mathbf{g}
\end{aligned}$$

Simplify the fourth term.

$$\begin{aligned}
\mu\nabla\cdot(\nabla\mathbf{v})^\dagger &= \mu\left(\sum_{i=1}^3 \boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left[\left(\sum_{j=1}^3 \boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\left(\sum_{k=1}^3 \boldsymbol{\delta}_k v_k\right)\right]^\dagger \\
&= \mu\left(\sum_{i=1}^3 \boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_{j=1}^3\sum_{k=1}^3 \boldsymbol{\delta}_j\boldsymbol{\delta}_k\frac{\partial v_k}{\partial x_j}\right)^\dagger \\
&= \mu\left(\sum_{i=1}^3 \boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_{j=1}^3\sum_{k=1}^3 \boldsymbol{\delta}_j\boldsymbol{\delta}_k\frac{\partial v_j}{\partial x_k}\right) \\
&= \mu\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)\boldsymbol{\delta}_k\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_k}\right) \\
&= \mu\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\delta_{ij}\boldsymbol{\delta}_k\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_k}\right) \\
&= \mu\sum_{j=1}^3\sum_{k=1}^3\boldsymbol{\delta}_k\frac{\partial}{\partial x_j}\left(\frac{\partial v_j}{\partial x_k}\right) \\
&= \mu\sum_{j=1}^3\sum_{k=1}^3\boldsymbol{\delta}_k\frac{\partial}{\partial x_k}\left(\frac{\partial v_j}{\partial x_j}\right) \\
&= \mu\sum_{k=1}^3\boldsymbol{\delta}_k\frac{\partial}{\partial x_k}\left(\sum_{j=1}^3\frac{\partial v_j}{\partial x_j}\right) \\
&= \mu\nabla(\nabla\cdot\mathbf{v})
\end{aligned}$$

The equation of motion then becomes

$$\frac{\partial}{\partial t}\rho\mathbf{v} = -\nabla\cdot\rho\mathbf{v}\mathbf{v} - \nabla p + \mu\nabla^2\mathbf{v} + \mu\nabla(\nabla\cdot\mathbf{v}) - \left(\frac{2}{3}\mu - \kappa\right)\nabla\cdot[(\nabla\cdot\mathbf{v})\boldsymbol{\delta}] + \rho\mathbf{g}.$$

With the assumption that the mass density ρ is constant, the continuity equation reduces to

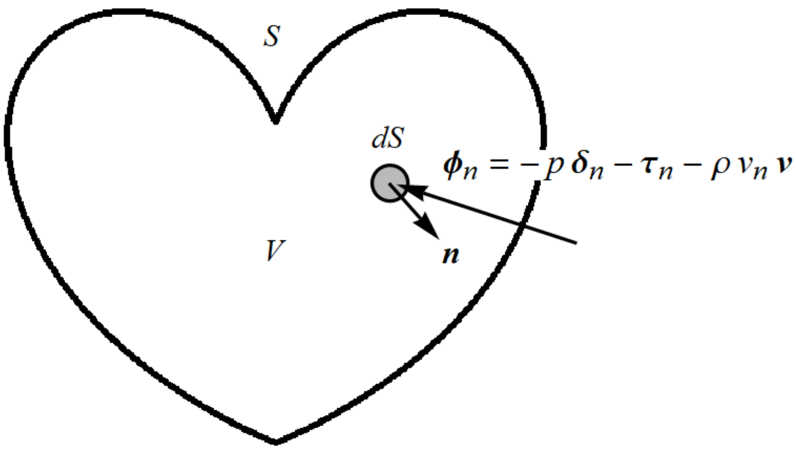
$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v} \quad \rightarrow \quad 0 = -\rho \nabla \cdot \mathbf{v} \quad \rightarrow \quad \nabla \cdot \mathbf{v} = 0,$$

and the legendary Navier-Stokes equation is obtained.

$$\frac{\partial}{\partial t}\rho \mathbf{v} + \nabla \cdot \rho \mathbf{v}\mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}.$$

Part (b)

Consider now a fixed arbitrary three-dimensional control volume with constant volume V and surface area S through which a fluid flows. Let $\mathbf{n}$ be the outward unit vector normal to the surface at every point.



Make a momentum balance over this control volume.

$$\begin{bmatrix}\text{rate of momentum increase} \\ \text{with respect to time}\end{bmatrix} = \begin{bmatrix}\text{rate of momentum} \\ \text{in}\end{bmatrix} - \begin{bmatrix}\text{rate of momentum} \\ \text{out}\end{bmatrix} + \begin{bmatrix}\text{external} \\ \text{forces}\end{bmatrix}$$

Gravity is the only external force; to get the total force, integrate $\mathbf{g}$ over the mass within the control volume. In general, the combined momentum flux $\boldsymbol{\phi} = \boldsymbol{\pi} + \rho \mathbf{v}\mathbf{v} = p\boldsymbol{\delta} + \boldsymbol{\tau} + \rho \mathbf{v}\mathbf{v}$ is a second-order tensor that, when integrated over the control volume's boundary, gives the rate of momentum in minus the rate of momentum out. This new term $\rho \mathbf{v}\mathbf{v}$ is the momentum flux due to convection of the fluid.

$$\frac{d\mathbf{p}}{dt} = \int_S \boldsymbol{\phi} \cdot d\mathbf{S} + \int_{\text{c.v. mass}} \mathbf{g}\, dm$$

Momentum that's being transported into the control volume is antiparallel to $\mathbf{n}$, making "rate of momentum in" negative, and momentum that's being transported out of the control volume is parallel to $\mathbf{n}$, making "rate of momentum out" positive. In other words, the dot product yields a minus sign.

$$\frac{d\mathbf{p}}{dt} = \int_S (-\boldsymbol{\phi}) \cdot (\mathbf{n}\, dS) + \int_{\text{c.v. mass}} \mathbf{g}\, dm$$

$\mathbf{p}$ is the linear momentum, and the total is obtained by summing (integrating) all the momenta in the control volume.

$$\frac{d}{dt} \int_{\text{c.v. mass}} \mathbf{v}\, dm = -\int_S \boldsymbol{\phi} \cdot \mathbf{n}\, dS + \int_{\text{c.v. mass}} \mathbf{g}\, dm$$

Introduce the mass density ρ to turn these mass integrals into volume integrals: $dm = \rho\, dV$.

$$\frac{d}{dt}\int_V \rho\mathbf{v}\, dV = -\int_S \boldsymbol{\phi}\cdot\mathbf{n}\, dS + \int_V \rho\mathbf{g}\, dV$$

Use Gauss's theorem to turn the surface integral into a volume integral as well.

$$\frac{d}{dt}\int_V \rho\mathbf{v}\, dV = -\int_V \nabla\cdot\boldsymbol{\phi}\, dV + \int_V \rho\mathbf{g}\, dV$$

Since V is constant, the time derivative can be brought inside the integral.

$$\int_V \frac{\partial}{\partial t}\rho\mathbf{v}\, dV = -\int_V \nabla\cdot\boldsymbol{\phi}\, dV + \int_V \rho\mathbf{g}\, dV$$

The control volume is arbitrary, so the volume integrals may be removed.

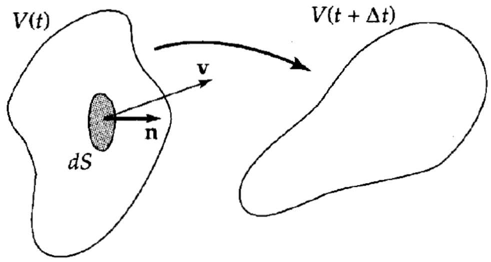
$$\begin{aligned}\frac{\partial}{\partial t}\rho\mathbf{v} &= -\nabla\cdot\boldsymbol{\phi} + \rho\mathbf{g}\\ &= -\nabla\cdot(p\boldsymbol{\delta} + \boldsymbol{\tau} + \rho\mathbf{v}\mathbf{v}) + \rho\mathbf{g}\\ &= -\nabla\cdot(p\boldsymbol{\delta}) - \nabla\cdot\boldsymbol{\tau} - \nabla\cdot\rho\mathbf{v}\mathbf{v} + \rho\mathbf{g}\\ &= -\nabla p - \nabla\cdot\boldsymbol{\tau} - \nabla\cdot\rho\mathbf{v}\mathbf{v} + \rho\mathbf{g}\end{aligned}$$

As shown in part (a), plugging in Newton's generalized law of viscosity for $\boldsymbol{\tau}$ and assuming constant density ρ and viscosity μ results in the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}.$$

Part (c)

Consider again the moving blob shown in Fig. 3D.1 with volume $V(t)$ and surface area $S(t)$.



Make a mass balance over this blob.

$$\begin{bmatrix}\text{rate of mass increase}\\\text{with respect to time}\end{bmatrix} = \begin{bmatrix}\text{rate of mass}\\\text{in}\end{bmatrix} - \begin{bmatrix}\text{rate of mass}\\\text{out}\end{bmatrix} + \begin{bmatrix}\text{rate of mass}\\\text{accumulation}\end{bmatrix}$$

Since the boundary of the blob moves with the fluid's velocity, it's impossible for fluid to actually enter or exit the blob. As a result, the rate of mass in and the rate of mass out are zero. By the law of conservation of mass, mass is neither created nor destroyed, so the rate of accumulation is zero as well.

$$\frac{dm}{dt} = 0$$

The blob mass is obtained by integrating the density ρ over the volume $V(t)$.

$$\frac{d}{dt}\int_{V(t)} \rho\, dV = 0$$

Apply the three-dimensional analog of the Leibnitz rule (formula A.5-5 on page 824) to differentiate the integral.

$$\int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \int_{S(t)} \rho(\mathbf{v}_S \cdot \mathbf{n})\, dS = 0$$

The blob surface moves with the fluid, so $\mathbf{v}_S = \mathbf{v}$.

$$\int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \int_{S(t)} \rho(\mathbf{v} \cdot \mathbf{n})\, dS = 0$$

Remove the parentheses.

$$\int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \int_{S(t)} \rho\mathbf{v} \cdot \mathbf{n}\, dS = 0$$

Apply Gauss's theorem to turn the surface integral into a volume integral.

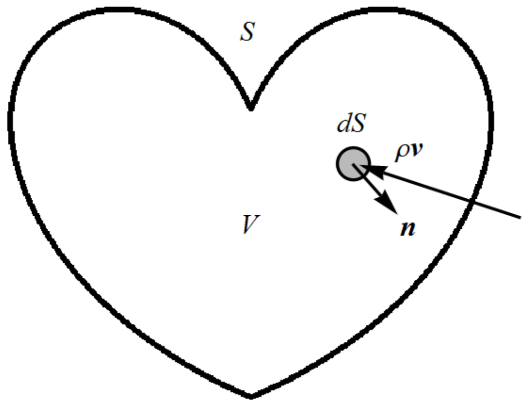
$$\int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \int_{V(t)} \nabla \cdot \rho\mathbf{v}\, dV = 0$$

Since the choice of the blob is arbitrary, all the integrals may be removed, resulting in the continuity equation.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\mathbf{v} = 0$$

Part (d)

Consider again a fixed three-dimensional arbitrary control volume with constant volume V and surface area S through which a fluid flows.



Make a mass balance over this control volume.

$$\begin{bmatrix}\text{rate of mass increase}\\ \text{with respect to time}\end{bmatrix} = \begin{bmatrix}\text{rate of mass}\\ \text{in}\end{bmatrix} - \begin{bmatrix}\text{rate of mass}\\ \text{out}\end{bmatrix} + \begin{bmatrix}\text{rate of mass}\\ \text{accumulation}\end{bmatrix}$$

A fluid with density ρ and velocity $\mathbf{v}$ carries a momentum $\rho\mathbf{v}$ per unit volume. Integrating $\rho\mathbf{v}$ over the surface of the control volume gives the rate that mass flows in minus the rate that mass flows out. By the law of conservation of mass, mass is neither created nor destroyed, so the rate of accumulation is zero.

$$\frac{dm}{dt} = \int_S \rho\mathbf{v} \cdot d\mathbf{S}$$

Mass that's being transported into the control volume is antiparallel to $\mathbf{n}$, making "rate of mass in" negative, and mass that's being transported out of the control volume is parallel to $\mathbf{n}$, making "rate of mass out" positive. In other words, the dot product yields a minus sign.

$$\frac{dm}{dt} = \int_S (-\rho\mathbf{v}) \cdot (\mathbf{n}\, dS)$$

The mass within the control volume is obtained by integrating the density ρ over the volume V.

$$\frac{d}{dt}\int_V \rho\, dV = -\int_S \rho\mathbf{v} \cdot \mathbf{n}\, dS$$

V is constant, so the time derivative can be brought inside the integral.

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_S \rho\mathbf{v} \cdot \mathbf{n}\, dS$$

Apply Gauss's theorem to turn the surface integral into a volume integral.

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_V \nabla \cdot \rho\mathbf{v}\, dV$$

Since the choice of the control volume is arbitrary, all the integrals may be removed, resulting in the continuity equation.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho\mathbf{v}$$