Problem 3D.2

The equation of change for vorticity.

(a) By taking the curl of the Navier-Stokes equation of motion (in either the D/Dt form or the $\partial/\partial t$ form), obtain an equation for the *vorticity*, $\mathbf{w} = [\nabla \times \mathbf{v}]$ of the fluid; this equation may be written in two ways:

$$\frac{D}{Dt}\mathbf{w} = \nu\nabla^2\mathbf{w} + [\mathbf{w}\cdot\nabla\mathbf{v}] \tag{3D.2-1}$$

$$\frac{D}{Dt}\mathbf{w} = \nu\nabla^2\mathbf{w} + [\boldsymbol{\varepsilon} : \{(\nabla\mathbf{v})\cdot(\nabla\mathbf{v})\}] \tag{3D.2-2}$$

in which $\boldsymbol{\varepsilon}$ is a third-order tensor whose components are the permutation symbol ε_{ijk} (see §A.2) and $\nu = \mu/\rho$ is the kinematic viscosity.

(b) How do the equations in (a) simplify for two-dimensional flows?

Solution

Part (a)

In terms of the substantial derivative, the Navier-Stokes equation is

$$\frac{D}{Dt}\rho\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}.$$

It holds under the assumption that the fluid density ρ and viscosity μ are constant. This equation can be written in one of two ways,

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \mathbf{v}\cdot\nabla\rho\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$
$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g},$$

and they will lead to Eq. 3D.2-1 and Eq. 3D.2-2, respectively. The force of gravity is conservative, so there exists a potential function Φ such that $m\mathbf{g} = -\nabla\Phi$.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \mathbf{v}\cdot\nabla\rho\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} - \frac{\rho}{m}\nabla\Phi$$
$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} - \frac{\rho}{m}\nabla\Phi$$

In order to eliminate the terms involving pressure and gravity, take the curl of both sides of each equation.

$$\nabla\times\left(\frac{\partial}{\partial t}\rho\mathbf{v} + \mathbf{v}\cdot\nabla\rho\mathbf{v}\right) = \nabla\times\left(-\nabla p + \mu\nabla^2\mathbf{v} - \frac{\rho}{m}\nabla\Phi\right)$$
$$\nabla\times\left(\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v}\right) = \nabla\times\left(-\nabla p + \mu\nabla^2\mathbf{v} - \frac{\rho}{m}\nabla\Phi\right)$$

The curl of a sum is the sum of the curls.

$$\nabla \times \left(\frac{\partial}{\partial t}\rho\mathbf{v}\right) + \nabla \times (\mathbf{v}\cdot\nabla\rho\mathbf{v}) = \nabla \times (-\nabla p) + \nabla \times (\mu\nabla^2\mathbf{v}) + \nabla \times \left(-\frac{\rho}{m}\nabla\Phi\right)$$
$$\nabla \times \left(\frac{\partial}{\partial t}\rho\mathbf{v}\right) + \nabla \times (\nabla\cdot\rho\mathbf{v}\mathbf{v}) = \nabla \times (-\nabla p) + \nabla \times (\mu\nabla^2\mathbf{v}) + \nabla \times \left(-\frac{\rho}{m}\nabla\Phi\right)$$

Bring the constants in front.

$$\rho\nabla \times \left(\frac{\partial}{\partial t}\mathbf{v}\right) + \rho\nabla \times (\mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla\times\nabla p + \mu\nabla \times (\nabla^2\mathbf{v}) - \frac{\rho}{m}\nabla\times\nabla\Phi$$
$$\rho\nabla \times \left(\frac{\partial}{\partial t}\mathbf{v}\right) + \rho\nabla \times (\nabla\cdot\mathbf{v}\mathbf{v}) = -\nabla\times\nabla p + \mu\nabla \times (\nabla^2\mathbf{v}) - \frac{\rho}{m}\nabla\times\nabla\Phi$$

Divide both sides of each equation by ρ and use the kinematic viscosity ν for μ/ρ.

$$\nabla \times \left(\frac{\partial}{\partial t}\mathbf{v}\right) + \nabla \times (\mathbf{v}\cdot\nabla\mathbf{v}) = -\frac{1}{\rho}\nabla\times\nabla p + \nu\nabla \times (\nabla^2\mathbf{v}) - \frac{1}{m}\nabla\times\nabla\Phi \tag{1}$$
$$\nabla \times \left(\frac{\partial}{\partial t}\mathbf{v}\right) + \nabla \times (\nabla\cdot\mathbf{v}\mathbf{v}) = -\frac{1}{\rho}\nabla\times\nabla p + \nu\nabla \times (\nabla^2\mathbf{v}) - \frac{1}{m}\nabla\times\nabla\Phi \tag{2}$$

Now examine each of the common terms in these equations one by one.

$$\begin{aligned}
\nabla\times\nabla p &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)p \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_j)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}p \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}p \\
&= \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\varepsilon_{jik}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}p \qquad \text{(Let } i \text{ be } j \text{ and let } j \text{ be } i.\text{)} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\varepsilon_{jik}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}p \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\varepsilon_{jik}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}p \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k(-\varepsilon_{ijk})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}p \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}p \\
&= \mathbf{0} \\
&= \nabla\times\nabla\Phi
\end{aligned}$$

For future reference, know that the curl of the gradient of a scalar function is the zero vector, provided that this function has continuous second partial derivatives.

$$
\begin{aligned}
\nabla \times \left(\frac{\partial}{\partial t}\mathbf{v}\right) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\frac{\partial}{\partial t}\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial v_j}{\partial t}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial t} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial t} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial}{\partial t}\frac{\partial v_j}{\partial x_i} \\
&= \frac{\partial}{\partial t}\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial}{\partial x_i} v_j\right) \\
&= \frac{\partial}{\partial t}(\nabla \times \mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times (\nabla^2 \mathbf{v}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}\left(\sum_{l=1}^{3} \boldsymbol{\delta}_l v_l\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \frac{\partial^2 v_l}{\partial x_j^2}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_l) \frac{\partial}{\partial x_i}\frac{\partial^2 v_l}{\partial x_j^2} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_i}\frac{\partial^2 v_l}{\partial x_j^2} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial^2}{\partial x_j^2}\frac{\partial v_l}{\partial x_i} \\
&= \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}\left(\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_i} v_l\right) \\
&= \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}(\nabla \times \mathbf{v}) \\
&= \nabla^2(\nabla \times \mathbf{v})
\end{aligned}
$$

With these results, equations (1) and (2) become

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) + \nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) = \nu \nabla^2 (\nabla \times \mathbf{v}) \tag{3}$$

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) + \nabla \times (\nabla \cdot \mathbf{v}\mathbf{v}) = \nu \nabla^2 (\nabla \times \mathbf{v}). \tag{4}$$

Use the vector identity,

$$\begin{aligned}
\mathbf{v} \times (\nabla \times \mathbf{v}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i v_i\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k v_k\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i v_i\right) \times \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) \frac{\partial v_k}{\partial x_j}\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i v_i\right) \times \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} \frac{\partial v_k}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_l) \varepsilon_{jkl} v_i \frac{\partial v_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \varepsilon_{jkl} v_i \frac{\partial v_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{mil} \varepsilon_{jkl} v_i \frac{\partial v_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m (\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}) v_i \frac{\partial v_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \delta_{mj}\delta_{ik} v_i \frac{\partial v_k}{\partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \delta_{mk}\delta_{ij} v_i \frac{\partial v_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \delta_{ik} v_i \frac{\partial v_k}{\partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \delta_{ij} v_i \frac{\partial v_k}{\partial x_j} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j v_k \frac{\partial v_k}{\partial x_j} - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k v_j \frac{\partial v_k}{\partial x_j} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2} v_k^2\right) - \sum_{j=1}^{3} v_j \frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3} \boldsymbol{\delta}_k v_k\right) \\
&= \frac{1}{2}\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3} v_k^2\right) - \sum_{j=1}^{3} v_j \frac{\partial}{\partial x_j}\mathbf{v} \\
&= \frac{1}{2}\nabla v^2 - \mathbf{v} \cdot \nabla \mathbf{v},
\end{aligned}$$

to rewrite the second term on the left side of equation (3).

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) + \nabla \times \left[\frac{1}{2}\nabla v^2 - \mathbf{v} \times (\nabla \times \mathbf{v})\right] = \nu \nabla^2 (\nabla \times \mathbf{v})$$

Distribute the curl operator and bring the constants in front.

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) + \frac{1}{2}\nabla \times \nabla v^2 - \nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})] = \nu \nabla^2 (\nabla \times \mathbf{v})$$

The curl of the gradient of a scalar function is the zero vector.

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) - \nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})] = \nu \nabla^2 (\nabla \times \mathbf{v}) \qquad \text{(Eq. 4.2-1)}$$

Let the vorticity be $\mathbf{w} = \nabla \times \mathbf{v}$.

$$\frac{\partial}{\partial t}\mathbf{w} - \nabla \times (\mathbf{v} \times \mathbf{w}) = \nu \nabla^2 \mathbf{w} \qquad (5)$$

Simplify the remaining cross product.

$$\begin{aligned}
\nabla \times (\mathbf{v} \times \mathbf{w}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k w_k\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) v_j w_k\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} v_j w_k\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} v_j w_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \varepsilon_{jkl} \frac{\partial}{\partial x_i} v_j w_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{mil} \varepsilon_{jkl} \frac{\partial}{\partial x_i} v_j w_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m (\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}) \frac{\partial}{\partial x_i} v_j w_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \delta_{mj}\delta_{ik} \frac{\partial}{\partial x_i} v_j w_k - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \delta_{mk}\delta_{ij} \frac{\partial}{\partial x_i} v_j w_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \delta_{ik} \frac{\partial}{\partial x_i} v_j w_k - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \delta_{ij} \frac{\partial}{\partial x_i} v_j w_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_i} v_j w_i - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_j} v_j w_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_j \left(\frac{\partial v_j}{\partial x_i} w_i + v_j \frac{\partial w_i}{\partial x_i}\right) - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \left(\frac{\partial v_j}{\partial x_j} w_k + v_j \frac{\partial w_k}{\partial x_j}\right)
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\nabla \times (\mathbf{v} \times \mathbf{w}) &= \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_j w_i \frac{\partial v_j}{\partial x_i} + \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j \frac{\partial w_i}{\partial x_i} - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial v_j}{\partial x_j} w_k - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k v_j \frac{\partial w_k}{\partial x_j} \\
&= \sum_{i=1}^{3} w_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j\right) + \sum_{j=1}^{3} \boldsymbol{\delta}_j v_j \left(\sum_{i=1}^{3} \frac{\partial w_i}{\partial x_i}\right) - \sum_{k=1}^{3} \boldsymbol{\delta}_k w_k \left(\sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j}\right) - \sum_{j=1}^{3} v_j \frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3} \boldsymbol{\delta}_k w_k\right) \\
&= \mathbf{w} \cdot \nabla \mathbf{v} + \mathbf{v}(\nabla \cdot \mathbf{w}) - \mathbf{w}(\nabla \cdot \mathbf{v}) - \mathbf{v} \cdot \nabla \mathbf{w}
\end{aligned}$$

For the particular case that $\mathbf{w} = \nabla \times \mathbf{v}$, the second term vanishes.

$$\begin{aligned}
\nabla \cdot \mathbf{w} &= \nabla \cdot (\nabla \times \mathbf{v}) \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k v_k\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) \frac{\partial}{\partial x_j} v_k\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} \frac{\partial v_k}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial v_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{il} \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial v_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jki} \frac{\partial}{\partial x_i} \frac{\partial v_k}{\partial x_j} \\
&= \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3} \varepsilon_{ikj} \frac{\partial}{\partial x_j} \frac{\partial v_k}{\partial x_i} \qquad \text{(Let } i \text{ be } j \text{ and let } j \text{ be } i.\text{)} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ikj} \frac{\partial}{\partial x_j} \frac{\partial v_k}{\partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ikj} \frac{\partial}{\partial x_i} \frac{\partial v_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} (-\varepsilon_{jki}) \frac{\partial}{\partial x_i} \frac{\partial v_k}{\partial x_j} \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jki} \frac{\partial}{\partial x_i} \frac{\partial v_k}{\partial x_j} \\
&= 0
\end{aligned}$$

As a result, equation (5) becomes

$$\frac{\partial}{\partial t}\mathbf{w} - [\mathbf{w}\cdot\nabla\mathbf{v} + \underbrace{\mathbf{v}(\nabla\cdot\mathbf{w})}_{=\mathbf{0}} - \mathbf{w}(\nabla\cdot\mathbf{v}) - \mathbf{v}\cdot\nabla\mathbf{w}] = \nu\nabla^2\mathbf{w}.$$

Because the density is constant, the continuity equation reduces to

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\rho\mathbf{v} \quad\rightarrow\quad 0 = -\rho\nabla\cdot\mathbf{v} \quad\rightarrow\quad \nabla\cdot\mathbf{v} = 0,$$

and this previous equation simplifies further to

$$\frac{\partial}{\partial t}\mathbf{w} - [\mathbf{w}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{w}] = \nu\nabla^2\mathbf{w}.$$

Therefore, one equation for the vorticity is

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{v}\cdot\nabla\mathbf{w} = \nu\nabla^2\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{v},$$

or

$$\boxed{\frac{D}{Dt}\mathbf{w} = \nu\nabla^2\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{v}.}$$

Now simplify the last term in equation (4).

$$\begin{aligned}
\nabla\times(\nabla\cdot\mathbf{v}\mathbf{v}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\cdot\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k v_k\right)\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l v_l\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\cdot\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_l v_k v_l\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)\boldsymbol{\delta}_l\frac{\partial}{\partial x_j}v_k v_l\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{jk}\boldsymbol{\delta}_l\frac{\partial}{\partial x_j}v_k v_l\right) \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left(\sum_{j=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial}{\partial x_j}v_j v_l\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_l)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}v_j v_l \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{ilm}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}v_j v_l \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{ilm}\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}v_l + v_j\frac{\partial v_l}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{ilm}\left[\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right)v_l + \frac{\partial v_j}{\partial x_j}\frac{\partial v_l}{\partial x_i} + \frac{\partial v_j}{\partial x_i}\frac{\partial v_l}{\partial x_j} + v_j\frac{\partial}{\partial x_i}\left(\frac{\partial v_l}{\partial x_j}\right)\right]
\end{aligned}$$

Continue simplifying the right side.

$$\begin{aligned}
\nabla \times (\nabla \cdot \mathbf{v}\mathbf{v}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right) v_l \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial v_j}{\partial x_j}\frac{\partial v_l}{\partial x_i} \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial v_j}{\partial x_i}\frac{\partial v_l}{\partial x_j} \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} v_j \frac{\partial}{\partial x_i}\left(\frac{\partial v_l}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_j}\right) v_l \\
&\quad + \left(\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_j}\right)\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial v_l}{\partial x_i} \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm}\delta_{in}\delta_{lp} \frac{\partial v_j}{\partial x_n}\frac{\partial v_p}{\partial x_j} \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} v_j \frac{\partial}{\partial x_j}\left(\frac{\partial v_l}{\partial x_i}\right) \\
&= \sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{v}) v_l \\
&\quad + (\nabla \cdot \mathbf{v})\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_i} v_l \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm}(\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_n)(\boldsymbol{\delta}_l \cdot \boldsymbol{\delta}_p) \frac{\partial v_j}{\partial x_n}\frac{\partial v_p}{\partial x_j} \\
&\quad + \sum_{j=1}^{3} v_j \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_i} v_l\right) \\
&= \nabla(\nabla \cdot \mathbf{v}) \times \mathbf{v} \\
&\quad + (\nabla \cdot \mathbf{v})(\nabla \times \mathbf{v}) \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm}(\boldsymbol{\delta}_l\boldsymbol{\delta}_i : \boldsymbol{\delta}_n\boldsymbol{\delta}_p) \frac{\partial v_j}{\partial x_n}\frac{\partial v_p}{\partial x_j} \\
&\quad + \mathbf{v} \cdot \nabla(\nabla \times \mathbf{v})
\end{aligned}$$

Continue simplifying the right side.

$$\begin{aligned}
\nabla \times (\nabla \cdot \mathbf{vv}) &= \nabla(\nabla \cdot \mathbf{v}) \times \mathbf{v} + (\nabla \cdot \mathbf{v})(\nabla \times \mathbf{v}) + \mathbf{v} \cdot \nabla(\nabla \times \mathbf{v}) \\
&\quad + \left(\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \varepsilon_{ilm}\boldsymbol{\delta}_m\boldsymbol{\delta}_l\boldsymbol{\delta}_i\right) : \left(\sum_{j=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3} \boldsymbol{\delta}_n\boldsymbol{\delta}_p \frac{\partial v_j}{\partial x_n}\frac{\partial v_p}{\partial x_j}\right) \\
&= \nabla(\nabla \cdot \mathbf{v}) \times \mathbf{v} + (\nabla \cdot \mathbf{v})(\nabla \times \mathbf{v}) + \mathbf{v} \cdot \nabla(\nabla \times \mathbf{v}) \\
&\quad + \left(\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \varepsilon_{mil}\boldsymbol{\delta}_m\boldsymbol{\delta}_l\boldsymbol{\delta}_i\right) : \left(\sum_{j=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3} \boldsymbol{\delta}_n\delta_{jq}\boldsymbol{\delta}_p \frac{\partial v_j}{\partial x_n}\frac{\partial v_p}{\partial x_q}\right) \\
&= \nabla(\nabla \cdot \mathbf{v}) \times \mathbf{v} + (\nabla \cdot \mathbf{v})(\nabla \times \mathbf{v}) + \mathbf{v} \cdot \nabla(\nabla \times \mathbf{v}) \\
&\quad + \left[\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} (-\varepsilon_{mli})\boldsymbol{\delta}_m\boldsymbol{\delta}_l\boldsymbol{\delta}_i\right] : \left[\sum_{n=1}^{3}\sum_{j=1}^{3}\sum_{q=1}^{3}\sum_{p=1}^{3} \boldsymbol{\delta}_n(\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_q)\boldsymbol{\delta}_p \frac{\partial v_j}{\partial x_n}\frac{\partial v_p}{\partial x_q}\right] \\
&= \nabla(\nabla \cdot \mathbf{v}) \times \mathbf{v} + (\nabla \cdot \mathbf{v})(\nabla \times \mathbf{v}) + \mathbf{v} \cdot \nabla(\nabla \times \mathbf{v}) \\
&\quad - \left(\sum_{m=1}^{3}\sum_{l=1}^{3}\sum_{i=1}^{3} \varepsilon_{mli}\boldsymbol{\delta}_m\boldsymbol{\delta}_l\boldsymbol{\delta}_i\right) : \left[\left(\sum_{n=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_n\boldsymbol{\delta}_j \frac{\partial v_j}{\partial x_n}\right) \cdot \left(\sum_{q=1}^{3}\sum_{p=1}^{3} \boldsymbol{\delta}_q\boldsymbol{\delta}_p \frac{\partial v_p}{\partial x_q}\right)\right] \\
&= \nabla(\nabla \cdot \mathbf{v}) \times \mathbf{v} + (\nabla \cdot \mathbf{v})(\nabla \times \mathbf{v}) + \mathbf{v} \cdot \nabla(\nabla \times \mathbf{v}) \\
&\quad - \left(\sum_{m=1}^{3}\sum_{l=1}^{3}\sum_{i=1}^{3} \varepsilon_{mli}\boldsymbol{\delta}_m\boldsymbol{\delta}_l\boldsymbol{\delta}_i\right) : \left[\left(\sum_{n=1}^{3} \boldsymbol{\delta}_n \frac{\partial}{\partial x_n}\right)\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j\right) \cdot \left(\sum_{q=1}^{3} \boldsymbol{\delta}_q \frac{\partial}{\partial x_q}\right)\left(\sum_{p=1}^{3} \boldsymbol{\delta}_p v_p\right)\right] \\
&= \nabla(\nabla \cdot \mathbf{v}) \times \mathbf{v} + (\nabla \cdot \mathbf{v})(\nabla \times \mathbf{v}) + \mathbf{v} \cdot \nabla(\nabla \times \mathbf{v}) - \boldsymbol{\varepsilon} : (\nabla\mathbf{v} \cdot \nabla\mathbf{v})
\end{aligned}$$

Substitute this result into equation (4).

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) + \nabla(\nabla \cdot \mathbf{v}) \times \mathbf{v} + (\nabla \cdot \mathbf{v})(\nabla \times \mathbf{v}) + \mathbf{v} \cdot \nabla(\nabla \times \mathbf{v}) - \boldsymbol{\varepsilon} : (\nabla\mathbf{v} \cdot \nabla\mathbf{v}) = \nu\nabla^2(\nabla \times \mathbf{v})$$

Because the density is constant, the continuity equation reduces to $\nabla \cdot \mathbf{v} = 0$, and this equation becomes

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) + \mathbf{v} \cdot \nabla(\nabla \times \mathbf{v}) - \boldsymbol{\varepsilon} : (\nabla\mathbf{v} \cdot \nabla\mathbf{v}) = \nu\nabla^2(\nabla \times \mathbf{v}).$$

Let the vorticity be $\mathbf{w} = \nabla \times \mathbf{v}$.

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{v} \cdot \nabla\mathbf{w} - \boldsymbol{\varepsilon} : (\nabla\mathbf{v} \cdot \nabla\mathbf{v}) = \nu\nabla^2\mathbf{w}$$

Therefore, a second equation for the vorticity is

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{v} \cdot \nabla\mathbf{w} = \nu\nabla^2\mathbf{w} + \boldsymbol{\varepsilon} : (\nabla\mathbf{v} \cdot \nabla\mathbf{v}),$$

or

$$\boxed{\frac{D}{Dt}\mathbf{w} = \nu\nabla^2\mathbf{w} + \boldsymbol{\varepsilon} : (\nabla\mathbf{v} \cdot \nabla\mathbf{v}).}$$

Part (b)

Suppose a fluid flows in an arbitrary plane in space. Setup a coordinate system so that the z-axis lies in the direction of the outward unit vector perpendicular to the plane. For a Cartesian coordinate system in particular, the velocity is then

$$\mathbf{v} = v_x(x, y, t)\hat{\mathbf{x}} + v_y(x, y, t)\hat{\mathbf{y}}.$$

Assuming that the fluid density ρ is constant, the equation of continuity simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

If the fluid viscosity μ is also constant, then the equation of motion simplifies to the Navier-Stokes equation. Take the curl of both sides of it and use the resulting vorticity equation (either form) instead.

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{v} \cdot \nabla \mathbf{w} = \nu \nabla^2 \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}$$

Similar to the Navier-Stokes equation, it is a vector equation. It actually represents three scalar equations, one for each variable in the chosen coordinate system. The vorticity is defined to be the curl of velocity.

$$\mathbf{w} = \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & 0 \end{vmatrix} = -\frac{\partial v_y}{\partial z}\hat{\mathbf{x}} + \frac{\partial v_x}{\partial z}\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Expand the continuity equation in Cartesian coordinates.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0$$

$$-\frac{\partial v_x}{\partial x} = \frac{\partial v_y}{\partial y}$$

If we introduce a stream function $\psi = \psi(x, y, t)$ that satisfies

$$v_x = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y = \frac{\partial \psi}{\partial x},$$

then the continuity equation will automatically be satisfied since the mixed partial derivatives are equal by Clairaut's theorem.

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$$

With this choice of v_x and v_y, the vorticity becomes

$$\mathbf{w} = \left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right]\hat{\mathbf{z}} = \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)\hat{\mathbf{z}} = (\nabla^2 \psi)\hat{\mathbf{z}}.$$

Substitute this expression for $\mathbf{w}$ into the vorticity equation.

$$\frac{\partial}{\partial t}[(\nabla^2 \psi)\hat{\mathbf{z}}] + \mathbf{v} \cdot \nabla[(\nabla^2 \psi)\hat{\mathbf{z}}] = \nu \nabla^2[(\nabla^2 \psi)\hat{\mathbf{z}}] + [(\nabla^2 \psi)\hat{\mathbf{z}}] \cdot \nabla \mathbf{v}$$

Expand the operators and simplify the equation.

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla^2\psi)\hat{\mathbf{z}} + v_x\frac{\partial}{\partial x}(\nabla^2\psi)\hat{\mathbf{z}} + v_y\frac{\partial}{\partial y}(\nabla^2\psi)\hat{\mathbf{z}} \\ + v_z\frac{\partial}{\partial z}(\nabla^2\psi)\hat{\mathbf{z}} = \nu(\nabla^4\psi)\hat{\mathbf{z}} + [(\nabla^2\psi)\hat{\mathbf{z}}]\cdot\bigg(\boldsymbol{\delta}_x\boldsymbol{\delta}_x\frac{\partial v_x}{\partial x} + \boldsymbol{\delta}_x\boldsymbol{\delta}_y\frac{\partial v_y}{\partial x} + \boldsymbol{\delta}_x\boldsymbol{\delta}_z\frac{\partial v_z}{\partial x} \\ + \boldsymbol{\delta}_y\boldsymbol{\delta}_x\frac{\partial v_x}{\partial y} + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\frac{\partial v_y}{\partial y} + \boldsymbol{\delta}_y\boldsymbol{\delta}_z\frac{\partial v_z}{\partial y} \\ + \boldsymbol{\delta}_z\boldsymbol{\delta}_x\frac{\partial v_x}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_y\frac{\partial v_y}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\frac{\partial v_z}{\partial z}\bigg)\end{aligned}$$

$$\frac{\partial}{\partial t}(\nabla^2\psi)\hat{\mathbf{z}} + v_x\frac{\partial}{\partial x}(\nabla^2\psi)\hat{\mathbf{z}} + v_y\frac{\partial}{\partial y}(\nabla^2\psi)\hat{\mathbf{z}} = \nu(\nabla^4\psi)\hat{\mathbf{z}} + (\nabla^2\psi)\left(\boldsymbol{\delta}_x\frac{\partial v_x}{\partial z} + \boldsymbol{\delta}_y\frac{\partial v_y}{\partial z} + \boldsymbol{\delta}_z\frac{\partial v_z}{\partial z}\right)$$

The last term vanishes because there is no dependence on z.

$$\frac{\partial}{\partial t}(\nabla^2\psi)\hat{\mathbf{z}} + v_x\frac{\partial}{\partial x}(\nabla^2\psi)\hat{\mathbf{z}} + v_y\frac{\partial}{\partial y}(\nabla^2\psi)\hat{\mathbf{z}} = \nu(\nabla^4\psi)\hat{\mathbf{z}}$$

Dot both sides by $\hat{\mathbf{z}}$, the unit vector normal to the plane, to get a scalar equation for ψ.

$$\frac{\partial}{\partial t}(\nabla^2\psi) + v_x\frac{\partial}{\partial x}(\nabla^2\psi) + v_y\frac{\partial}{\partial y}(\nabla^2\psi) = \nu\nabla^4\psi$$

Now eliminate v_x and v_y.

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \left(-\frac{\partial\psi}{\partial y}\right)\frac{\partial}{\partial x}(\nabla^2\psi) + \left(\frac{\partial\psi}{\partial x}\right)\frac{\partial}{\partial y}(\nabla^2\psi) = \nu\nabla^4\psi$$

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) = \nu\nabla^4\psi$$

Notice that these last two terms on the left side can be expressed as a Jacobian.

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \begin{vmatrix}\frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} \\ \frac{\partial}{\partial x}(\nabla^2\psi) & \frac{\partial}{\partial y}(\nabla^2\psi)\end{vmatrix} = \nu\nabla^4\psi$$

Therefore, for a two-dimensional flow in Cartesian coordinates, the equation the stream function satisfies is

$$\boxed{\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)} = \nu\nabla^4\psi.}$$

This is equation (A) in Table 4.2-1 on page 123. Note that ∇^4 is known as the biharmonic operator, and in Cartesian coordinates it expands as

$$\begin{aligned}\nabla^4\psi = \nabla^2(\nabla^2\psi) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \\ &= \frac{\partial^4\psi}{\partial x^4} + \frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^2\partial x^2} + \frac{\partial^4\psi}{\partial y^4} = \frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4}.\end{aligned}$$

Suppose a fluid flows in an arbitrary plane in space. Setup a coordinate system so that the z-axis lies in the direction of the outward unit vector perpendicular to the plane. For a cylindrical coordinate system in particular, the velocity is then

$$\mathbf{v} = v_r(r,\theta,t)\hat{\mathbf{r}} + v_\theta(r,\theta,t)\hat{\boldsymbol{\theta}}.$$

Assuming that the fluid density ρ is constant, the equation of continuity simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

If the fluid viscosity μ is also constant, then the equation of motion simplifies to the Navier-Stokes equation. Take the curl of both sides of it and use the resulting vorticity equation (either form) instead.

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{v}\cdot\nabla\mathbf{w} = \nu\nabla^2\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{v}$$

Similar to the Navier-Stokes equation, it is a vector equation. It actually represents three scalar equations, one for each variable in the chosen coordinate system. The vorticity is defined to be the curl of velocity. All of the needed expansions in cylindrical coordinates are given on page 834.

$$\mathbf{w} = \nabla\times\mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\hat{\mathbf{z}} = \left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\hat{\mathbf{z}}$$

Expand the continuity equation in cylindrical coordinates.

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \underbrace{\frac{\partial v_z}{\partial z}}_{=\,0} = 0$$

$$-\frac{\partial}{\partial r}(rv_r) = \frac{\partial v_\theta}{\partial\theta}$$

If we introduce a stream function $\psi = \psi(r,\theta,t)$ that satisfies

$$v_r = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad v_\theta = \frac{\partial\psi}{\partial r},$$

then the continuity equation will automatically be satisfied since the mixed partial derivatives are equal by Clairaut's theorem.

$$\frac{\partial^2\psi}{\partial r\partial\theta} = \frac{\partial^2\psi}{\partial\theta\partial r}$$

With this choice of v_r and v_θ, the vorticity becomes

$$\mathbf{w} = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)\right]\hat{\mathbf{z}} = \left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)\hat{\mathbf{z}} = (\nabla^2\psi)\hat{\mathbf{z}}.$$

Substitute this expression for $\mathbf{w}$ into the vorticity equation.

$$\frac{\partial}{\partial t}[(\nabla^2\psi)\hat{\mathbf{z}}] + \mathbf{v}\cdot\nabla[(\nabla^2\psi)\hat{\mathbf{z}}] = \nu\nabla^2[(\nabla^2\psi)\hat{\mathbf{z}}] + [(\nabla^2\psi)\hat{\mathbf{z}}]\cdot\nabla\mathbf{v}$$

Expand the operators and simplify the equation.

$$\frac{\partial}{\partial t}(\nabla^2\psi)\hat{\mathbf{z}} + \left[v_r\frac{\partial}{\partial r}(\nabla^2\psi) + \frac{v_\theta}{r}\frac{\partial}{\partial\theta}(\nabla^2\psi)\right]\hat{\mathbf{z}} = \nu(\nabla^4\psi)\hat{\mathbf{z}} + \nabla^2\phi\left(\frac{\partial v_r}{\partial z}\right)\hat{\mathbf{r}} + \nabla^2\phi\left(\frac{\partial v_\theta}{\partial z}\right)\hat{\boldsymbol{\theta}} + \nabla^2\phi\left(\frac{\partial v_z}{\partial z}\right)\hat{\mathbf{z}}$$

The last three terms on the right side vanish because there is no dependence on z.

$$\frac{\partial}{\partial t}(\nabla^2\psi)\hat{\mathbf{z}} + \left[v_r\frac{\partial}{\partial r}(\nabla^2\psi) + \frac{v_\theta}{r}\frac{\partial}{\partial\theta}(\nabla^2\psi)\right]\hat{\mathbf{z}} = \nu(\nabla^4\psi)\hat{\mathbf{z}}$$

Dot both sides by $\hat{\mathbf{z}}$, the unit vector normal to the plane, to get a scalar equation for ψ.

$$\frac{\partial}{\partial t}(\nabla^2\psi) + v_r\frac{\partial}{\partial r}(\nabla^2\psi) + \frac{v_\theta}{r}\frac{\partial}{\partial\theta}(\nabla^2\psi) = \nu\nabla^4\psi$$

Now eliminate v_r and v_θ.

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \left(-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)\frac{\partial}{\partial r}(\nabla^2\psi) + \frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right)\frac{\partial}{\partial\theta}(\nabla^2\psi) = \nu\nabla^4\psi$$

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{1}{r}\left[\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}(\nabla^2\psi) - \frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}(\nabla^2\psi)\right] = \nu\nabla^4\psi$$

Notice that the quantity in square brackets can be expressed as a Jacobian.

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{1}{r}\begin{vmatrix} \frac{\partial\psi}{\partial r} & \frac{\partial\psi}{\partial\theta} \\ \frac{\partial}{\partial r}(\nabla^2\psi) & \frac{\partial}{\partial\theta}(\nabla^2\psi)\end{vmatrix} = \nu\nabla^4\psi$$

Therefore, for a two-dimensional flow in polar coordinates, the equation the stream function satisfies is

$$\boxed{\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{1}{r}\frac{\partial(\psi,\nabla^2\psi)}{\partial(r,\theta)} = \nu\nabla^4\psi.}$$

This is equation (B) in Table 4.2-1 on page 123. Note that ∇^4 is known as the biharmonic operator, and in polar coordinates it expands as

$$\begin{aligned}
\nabla^4\psi = \nabla^2(\nabla^2\psi) &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) \\
&= \frac{\partial^2}{\partial r^2}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) \\
&\quad + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) \\
&\quad\quad + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) \\
&= \frac{\partial}{\partial r}\left(\frac{\partial^3\psi}{\partial r^3} - \frac{1}{r^2}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} - \frac{2}{r^3}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial^3\psi}{\partial r\partial\theta^2}\right) \\
&\quad + \frac{1}{r}\left(\frac{\partial^3\psi}{\partial r^3} - \frac{1}{r^2}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} - \frac{2}{r^3}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial^3\psi}{\partial r\partial\theta^2}\right) \\
&\quad\quad + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial^3\psi}{\partial\theta\partial r^2} + \frac{1}{r}\frac{\partial^2\psi}{\partial\theta\partial r} + \frac{1}{r^2}\frac{\partial^3\psi}{\partial\theta^3}\right) \\
&= \left(\frac{\partial^4\psi}{\partial r^4} + \frac{2}{r^3}\frac{\partial\psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial^3\psi}{\partial r^3} + \frac{6}{r^4}\frac{\partial^2\psi}{\partial\theta^2} - \frac{2}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2} - \frac{2}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2} + \frac{1}{r^2}\frac{\partial^4\psi}{\partial r^2\partial\theta^2}\right) \\
&\quad + \left(\frac{1}{r}\frac{\partial^3\psi}{\partial r^3} - \frac{1}{r^3}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial r^2} - \frac{2}{r^4}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2}\right) + \left(\frac{1}{r^2}\frac{\partial^4\psi}{\partial\theta^2\partial r^2} + \frac{1}{r^3}\frac{\partial^3\psi}{\partial\theta^2\partial r} + \frac{1}{r^4}\frac{\partial^4\psi}{\partial\theta^4}\right) \\
&= \frac{\partial^4\psi}{\partial r^4} + \frac{2}{r}\frac{\partial^3\psi}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^3}\frac{\partial\psi}{\partial r} + \frac{2}{r^2}\frac{\partial^4\psi}{\partial r^2\partial\theta^2} - \frac{2}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2} + \frac{4}{r^4}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^4\psi}{\partial\theta^4}.
\end{aligned}$$

Part (c)

Suppose a fluid flows radially and vertically with respect to an arbitrary axis in space. Setup a cylindrical coordinate system so that the z-axis lies along the axis of symmetry. The velocity is then

$$\mathbf{v} = v_r(r, z, t)\hat{\mathbf{r}} + v_z(r, z, t)\hat{\mathbf{z}}.$$

Assuming that the fluid density ρ is constant, the equation of continuity simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

If the fluid viscosity μ is also constant, then the equation of motion simplifies to the Navier-Stokes equation. Take the curl of both sides of it and use the resulting vorticity equation (either form) instead.

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{v} \cdot \nabla\mathbf{w} = \nu\nabla^2\mathbf{w} + \mathbf{w} \cdot \nabla\mathbf{v}$$

Similar to the Navier-Stokes equation, it is a vector equation. It actually represents three scalar equations, one for each variable in the chosen coordinate system. The vorticity is defined to be the curl of velocity. All of the needed expansions in cylindrical coordinates are given on page 834.

$$\mathbf{w} = \nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]\hat{\mathbf{z}} = \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\theta}}$$

Expand the continuity equation in cylindrical coordinates.

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=\,0} + \frac{\partial v_z}{\partial z} = 0$$

$$\frac{\partial}{\partial r}(rv_r) = -r\frac{\partial v_z}{\partial z}$$

If we introduce a stream function $\psi = \psi(r, z, t)$ that satisfies

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial z} \quad \text{and} \quad v_z = -\frac{1}{r}\frac{\partial \psi}{\partial r},$$

then the continuity equation will automatically be satisfied since the mixed partial derivatives are equal by Clairaut's theorem.

$$\frac{\partial^2 \psi}{\partial r \partial z} = \frac{\partial^2 \psi}{\partial z \partial r}$$

With this choice of v_r and v_z, the vorticity becomes

$$\mathbf{w} = \left[\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial z}\right) - \frac{\partial}{\partial r}\left(-\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\right]\hat{\boldsymbol{\theta}} = \frac{1}{r}\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}\right)\hat{\boldsymbol{\theta}} = \frac{1}{r}(E^2\psi)\hat{\boldsymbol{\theta}}.$$

Substitute this expression for $\mathbf{w}$ into the vorticity equation.

$$\frac{\partial}{\partial t}\left[\frac{1}{r}(E^2\psi)\hat{\boldsymbol{\theta}}\right] + \mathbf{v} \cdot \nabla\left[\frac{1}{r}(E^2\psi)\hat{\boldsymbol{\theta}}\right] = \nu\nabla^2\left[\frac{1}{r}(E^2\psi)\hat{\boldsymbol{\theta}}\right] + \left[\frac{1}{r}(E^2\psi)\hat{\boldsymbol{\theta}}\right] \cdot \nabla\mathbf{v}$$

The first operator on the left side simplifies like so.

$$\frac{\partial}{\partial t}\left[\frac{1}{r}(E^2\psi)\hat{\boldsymbol{\theta}}\right] = \frac{1}{r}\frac{\partial}{\partial t}(E^2\psi)\hat{\boldsymbol{\theta}}$$

The second operator on the left side expands like so (use equation (Q) on page 834).

$$\begin{aligned}
\mathbf{v}\cdot\nabla\left[\frac{1}{r}(E^2\psi)\hat{\boldsymbol{\theta}}\right] &= \left\{v_r\frac{\partial}{\partial r}\left[\frac{1}{r}(E^2\psi)\right] + v_z\frac{\partial}{\partial z}\left[\frac{1}{r}(E^2\psi)\right]\right\}\hat{\boldsymbol{\theta}} \\
&= \left\{v_r\left[-\frac{1}{r^2}(E^2\psi) + \frac{1}{r}\frac{\partial}{\partial r}(E^2\psi)\right] + v_z\left[\frac{1}{r}\frac{\partial}{\partial z}(E^2\psi)\right]\right\}\hat{\boldsymbol{\theta}} \\
&= -\frac{v_r}{r^2}(E^2\psi)\hat{\boldsymbol{\theta}} + \frac{v_r}{r}\frac{\partial}{\partial r}(E^2\psi)\hat{\boldsymbol{\theta}} + \frac{v_z}{r}\frac{\partial}{\partial z}(E^2\psi)\hat{\boldsymbol{\theta}}
\end{aligned}$$

The first operator on the right side expands like so (use equation (N) on page 834).

$$\begin{aligned}
\nu\nabla^2\left[\frac{1}{r}(E^2\psi)\hat{\boldsymbol{\theta}}\right] &= \nu\left\{\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{1}{r}(E^2\psi)\right]\right\} + \frac{\partial^2}{\partial z^2}\left[\frac{1}{r}(E^2\psi)\right]\right\}\hat{\boldsymbol{\theta}} \\
&= \nu\left[-\frac{1}{r^2}\frac{\partial}{\partial r}(E^2\psi) + \frac{1}{r}\frac{\partial^2}{\partial r^2}(E^2\psi) + \frac{1}{r}\frac{\partial^2}{\partial z^2}(E^2\psi)\right]\hat{\boldsymbol{\theta}} \\
&= \frac{\nu}{r}\left[\frac{\partial^2}{\partial r^2}(E^2\psi) - \frac{1}{r}\frac{\partial}{\partial r}(E^2\psi) + \frac{\partial^2}{\partial z^2}(E^2\psi)\right]\hat{\boldsymbol{\theta}} \\
&= \frac{\nu}{r}E^2(E^2\psi)\hat{\boldsymbol{\theta}} \\
&= \frac{\nu}{r}(E^4\psi)\hat{\boldsymbol{\theta}}
\end{aligned}$$

The second operator on the right side expands like so (use equation (Q) on page 834).

$$\begin{aligned}
\left[\frac{1}{r}(E^2\psi)\hat{\boldsymbol{\theta}}\right]\cdot\nabla\mathbf{v} &= \left[\frac{1}{r}(E^2\psi)\right]\left(\frac{v_r}{r}\right)\hat{\boldsymbol{\theta}} \\
&= \frac{v_r}{r^2}(E^2\psi)\hat{\boldsymbol{\theta}}
\end{aligned}$$

With these results, the vorticity equation becomes

$$\frac{1}{r}\frac{\partial}{\partial t}(E^2\psi)\hat{\boldsymbol{\theta}} - \frac{v_r}{r^2}(E^2\psi)\hat{\boldsymbol{\theta}} + \frac{v_r}{r}\frac{\partial}{\partial r}(E^2\psi)\hat{\boldsymbol{\theta}} + \frac{v_z}{r}\frac{\partial}{\partial z}(E^2\psi)\hat{\boldsymbol{\theta}} = \frac{\nu}{r}(E^4\psi)\hat{\boldsymbol{\theta}} + \frac{v_r}{r^2}(E^2\psi)\hat{\boldsymbol{\theta}}.$$

Dot both sides by $\hat{\boldsymbol{\theta}}$ to get a scalar equation.

$$\frac{1}{r}\frac{\partial}{\partial t}(E^2\psi) - \frac{v_r}{r^2}(E^2\psi) + \frac{v_r}{r}\frac{\partial}{\partial r}(E^2\psi) + \frac{v_z}{r}\frac{\partial}{\partial z}(E^2\psi) = \frac{\nu}{r}E^4\psi + \frac{v_r}{r^2}(E^2\psi)$$

Combine like-terms.

$$\frac{1}{r}\frac{\partial}{\partial t}(E^2\psi) + \frac{v_r}{r}\frac{\partial}{\partial r}(E^2\psi) + \frac{v_z}{r}\frac{\partial}{\partial z}(E^2\psi) - \frac{2v_r}{r^2}(E^2\psi) = \frac{\nu}{r}E^4\psi$$

Multiply both sides by r.

$$\frac{\partial}{\partial t}(E^2\psi) + v_r\frac{\partial}{\partial r}(E^2\psi) + v_z\frac{\partial}{\partial z}(E^2\psi) - \frac{2v_r}{r}(E^2\psi) = \nu E^4\psi$$

Now eliminate v_r and v_z.

$$\frac{\partial}{\partial t}(E^2\psi) + \left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\frac{\partial}{\partial r}(E^2\psi) + \left(-\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\frac{\partial}{\partial z}(E^2\psi) - \frac{2}{r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)(E^2\psi) = \nu E^4\psi$$

Simplify the left side.

$$\frac{\partial}{\partial t}(E^2\psi) - \frac{1}{r}\left[\frac{\partial \psi}{\partial r}\frac{\partial}{\partial z}(E^2\psi) - \frac{\partial \psi}{\partial z}\frac{\partial}{\partial r}(E^2\psi)\right] - \frac{2}{r^2}\frac{\partial \psi}{\partial z}(E^2\psi) = \nu E^4\psi$$

Notice that the quantity in square brackets can be written as a Jacobian.

$$\frac{\partial}{\partial t}(E^2\psi) - \frac{1}{r}\begin{vmatrix} \frac{\partial \psi}{\partial r} & \frac{\partial \psi}{\partial z} \\ \frac{\partial}{\partial r}(E^2\psi) & \frac{\partial}{\partial z}(E^2\psi) \end{vmatrix} - \frac{2}{r^2}\frac{\partial \psi}{\partial z}(E^2\psi) = \nu E^4\psi$$

Therefore, for a cylindrical axisymmetric flow with no θ-component of velocity or angular dependence, the equation for the stream function is

$$\boxed{\frac{\partial}{\partial t}(E^2\psi) - \frac{1}{r}\frac{\partial(\psi, E^2\psi)}{\partial(r, z)} - \frac{2}{r^2}\frac{\partial \psi}{\partial z}(E^2\psi) = \nu E^4\psi.}$$

This is equation (C) in Table 4.2-1 on page 123. Note that $E^4\psi$ expands as follows.

$$\begin{aligned}
E^4\psi &= E^2(E^2\psi) \\
&= \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}\right) \\
&= \frac{\partial^2}{\partial r^2}\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}\right) \\
&\quad - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}\right) \\
&\quad + \frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}\right) \\
&= \frac{\partial}{\partial r}\left(\frac{\partial^3 \psi}{\partial r^3} + \frac{1}{r^2}\frac{\partial \psi}{\partial r} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^3 \psi}{\partial r \partial z^2}\right) \\
&\quad - \frac{1}{r}\left(\frac{\partial^3 \psi}{\partial r^3} + \frac{1}{r^2}\frac{\partial \psi}{\partial r} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^3 \psi}{\partial r \partial z^2}\right) \\
&\quad + \left(\frac{\partial^4 \psi}{\partial z^2 \partial r^2} - \frac{1}{r}\frac{\partial^3 \psi}{\partial z^2 \partial r} + \frac{\partial^4 \psi}{\partial z^4}\right) \\
&= \left(\frac{\partial^4 \psi}{\partial r^4} - \frac{2}{r^3}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial^3 \psi}{\partial r^3} + \frac{\partial^4 \psi}{\partial r^2 \partial z^2}\right) \\
&\quad - \left(\frac{1}{r}\frac{\partial^3 \psi}{\partial r^3} + \frac{1}{r^3}\frac{\partial \psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial^3 \psi}{\partial r \partial z^2}\right) \\
&\quad + \left(\frac{\partial^4 \psi}{\partial z^2 \partial r^2} - \frac{1}{r}\frac{\partial^3 \psi}{\partial z^2 \partial r} + \frac{\partial^4 \psi}{\partial z^4}\right) \\
&= \frac{\partial^4 \psi}{\partial r^4} - \frac{2}{r}\frac{\partial^3 \psi}{\partial r^3} + \frac{3}{r^2}\frac{\partial^2 \psi}{\partial r^2} - \frac{3}{r^3}\frac{\partial \psi}{\partial r} - \frac{2}{r}\frac{\partial^3 \psi}{\partial r \partial z^2} + 2\frac{\partial^4 \psi}{\partial r^2 \partial z^2} + \frac{\partial^4 \psi}{\partial z^4}
\end{aligned}$$

Suppose the velocity of a fluid has a radial and polar component at an arbitrary point in space and that these components have no azimuthal dependence along some axis through this point. Setup a spherical coordinate system at this point with the polar axis oriented along the axis of symmetry. The velocity is then

$$\mathbf{v} = v_r(r,\theta,t)\hat{\mathbf{r}} + v_\theta(r,\theta,t)\hat{\boldsymbol{\theta}}.$$

Assuming that the fluid density ρ is constant, the equation of continuity simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

If the fluid viscosity μ is also constant, then the equation of motion simplifies to the Navier-Stokes equation. Take the curl of both sides of it and use the resulting vorticity equation (either form) instead.

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{v}\cdot\nabla\mathbf{w} = \nu\nabla^2\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{v}$$

Similar to the Navier-Stokes equation, it is a vector equation. It actually represents three scalar equations, one for each variable in the chosen coordinate system. The vorticity is defined to be the curl of velocity. All of the needed expansions in spherical coordinates are given on page 836.

$$\begin{aligned}\mathbf{w} = \nabla\times\mathbf{v} &= \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rv_\phi)\right]\hat{\boldsymbol{\theta}} + \left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}\\ &= \left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}\end{aligned}$$

Expand the continuity equation in spherical coordinates.

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \underbrace{\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}}_{=0} = 0$$

Multiply both sides by $r^2\sin\theta$.

$$\sin\theta\frac{\partial}{\partial r}(r^2 v_r) + r\frac{\partial}{\partial\theta}(v_\theta\sin\theta) = 0$$

$$\frac{\partial}{\partial r}(v_r r^2\sin\theta) + \frac{\partial}{\partial\theta}(v_\theta r\sin\theta) = 0$$

If we introduce a stream function $\psi = \psi(r,\theta,t)$ that satisfies

$$v_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},$$

then the continuity equation will automatically be satisfied since the mixed partial derivatives are equal by Clairaut's theorem.

$$-\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{\partial^2\psi}{\partial\theta\partial r} = 0$$

With this choice of v_r and v_θ, the vorticity becomes

$$\begin{aligned}\mathbf{w} &= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(-\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right]\hat{\boldsymbol{\phi}} = \frac{1}{r\sin\theta}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right]\hat{\boldsymbol{\phi}}\\ &= \frac{1}{r\sin\theta}\left(\frac{\partial^2\psi}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)\hat{\boldsymbol{\phi}} = \frac{1}{r\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}}.\end{aligned}$$

Substitute this expression for **w** into the vorticity equation.

$$\frac{\partial}{\partial t}\left[\frac{1}{r\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}}\right]+\mathbf{v}\cdot\nabla\left[\frac{1}{r\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}}\right]=\nu\nabla^2\left[\frac{1}{r\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}}\right]+\left[\frac{1}{r\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}}\right]\cdot\nabla\mathbf{v}$$

The first operator on the left side simplifies like so.

$$\frac{\partial}{\partial t}\left[\frac{1}{r\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}}\right]=\frac{1}{r\sin\theta}\frac{\partial}{\partial t}(E^2\psi)\hat{\boldsymbol{\phi}}$$

The second operator on the left side expands like so (use equation (R) on page 836).

$$\begin{aligned}
\mathbf{v}\cdot\nabla\left[\frac{1}{r\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}}\right]&=\left\{v_r\frac{\partial}{\partial r}\left[\frac{1}{r\sin\theta}(E^2\psi)\right]+v_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r\sin\theta}(E^2\psi)\right]\right\}\hat{\boldsymbol{\phi}}\\
&=\left\{v_r\left[-\frac{1}{r^2\sin\theta}(E^2\psi)+\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(E^2\psi)\right]+\frac{v_\theta}{r}\left[-\frac{\cos\theta}{r\sin^2\theta}(E^2\psi)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E^2\psi)\right]\right\}\hat{\boldsymbol{\phi}}\\
&=-\frac{v_r}{r^2\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}}+\frac{v_r}{r\sin\theta}\frac{\partial}{\partial r}(E^2\psi)\hat{\boldsymbol{\phi}}-\frac{v_\theta\cos\theta}{r^2\sin^2\theta}(E^2\psi)\hat{\boldsymbol{\phi}}+\frac{v_\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}(E^2\psi)\hat{\boldsymbol{\phi}}
\end{aligned}$$

The first operator on the right side expands like so (use equation (O) on page 836).

$$\begin{aligned}
\nu\nabla^2\left[\frac{1}{r\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}}\right]&=\nu\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}(E^2\psi)\right)\right]+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}(E^2\psi)\sin\theta\right)\right]\right\}\hat{\boldsymbol{\phi}}\\
&=\nu\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\frac{1}{r^2\sin\theta}(E^2\psi)+\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(E^2\psi)\right)\right]\right.\\
&\qquad\left.+\frac{1}{r^2}\left[-\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r}(E^2\psi)\right)+\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta^2}\left(\frac{1}{r}(E^2\psi)\right)\right]\right\}\hat{\boldsymbol{\phi}}\\
&=\nu\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[-\frac{1}{\sin\theta}(E^2\psi)+\frac{r}{\sin\theta}\frac{\partial}{\partial r}(E^2\psi)\right]\right.\\
&\qquad\left.+\frac{1}{r^2}\left[-\frac{\cos\theta}{\sin^2\theta}\frac{1}{r}\frac{\partial}{\partial\theta}(E^2\psi)+\frac{1}{\sin\theta}\frac{1}{r}\frac{\partial^2}{\partial\theta^2}(E^2\psi)\right]\right\}\hat{\boldsymbol{\phi}}\\
&=\nu\left\{\frac{1}{r^2}\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial r}(E^2\psi)+\frac{1}{\sin\theta}\frac{\partial}{\partial r}(E^2\psi)+\frac{r}{\sin\theta}\frac{\partial^2}{\partial r^2}(E^2\psi)\right]\right.\\
&\qquad\left.+\frac{1}{r^2}\left[-\frac{\cos\theta}{\sin^2\theta}\frac{1}{r}\frac{\partial}{\partial\theta}(E^2\psi)+\frac{1}{\sin\theta}\frac{1}{r}\frac{\partial^2}{\partial\theta^2}(E^2\psi)\right]\right\}\hat{\boldsymbol{\phi}}\\
&=\nu\left[\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r^2}(E^2\psi)-\frac{\cos\theta}{r^3\sin^2\theta}\frac{\partial}{\partial\theta}(E^2\psi)+\frac{1}{r^3\sin\theta}\frac{\partial^2}{\partial\theta^2}(E^2\psi)\right]\hat{\boldsymbol{\phi}}\\
&=\frac{\nu}{r\sin\theta}\left[\frac{\partial^2}{\partial r^2}(E^2\psi)-\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}(E^2\psi)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(E^2\psi)\right]\hat{\boldsymbol{\phi}}\\
&=\frac{\nu}{r\sin\theta}E^2(E^2\psi)\hat{\boldsymbol{\phi}}\\
&=\frac{\nu}{r\sin\theta}(E^4\psi)\hat{\boldsymbol{\phi}}
\end{aligned}$$

The second operator on the right side expands like so (use equation (R) on page 836).

$$\begin{aligned}
\left[\frac{1}{r\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}}\right]\cdot\nabla\mathbf{v}&=\left[\frac{1}{r\sin\theta}(E^2\psi)\right]\left(\frac{v_r}{r}+\frac{v_\theta}{r}\cot\theta\right)\hat{\boldsymbol{\phi}}\\
&=\frac{v_r}{r^2\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}}+\frac{v_\theta\cos\theta}{r^2\sin^2\theta}(E^2\psi)\hat{\boldsymbol{\phi}}
\end{aligned}$$

With these results, the vorticity equation becomes

$$\begin{aligned}\frac{1}{r\sin\theta}\frac{\partial}{\partial t}(E^2\psi)\hat{\boldsymbol{\phi}} - \frac{v_r}{r^2\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}} + \frac{v_r}{r\sin\theta}\frac{\partial}{\partial r}(E^2\psi)\hat{\boldsymbol{\phi}} \\ - \frac{v_\theta\cos\theta}{r^2\sin^2\theta}(E^2\psi)\hat{\boldsymbol{\phi}} + \frac{v_\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}(E^2\psi)\hat{\boldsymbol{\phi}} \\ = \frac{\nu}{r\sin\theta}(E^4\psi)\hat{\boldsymbol{\phi}} + \frac{v_r}{r^2\sin\theta}(E^2\psi)\hat{\boldsymbol{\phi}} + \frac{v_\theta\cos\theta}{r^2\sin^2\theta}(E^2\psi)\hat{\boldsymbol{\phi}}.\end{aligned}$$

Dot both sides by $\hat{\boldsymbol{\phi}}$ to get a scalar equation.

$$\begin{aligned}\frac{1}{r\sin\theta}\frac{\partial}{\partial t}(E^2\psi) - \frac{v_r}{r^2\sin\theta}(E^2\psi) + \frac{v_r}{r\sin\theta}\frac{\partial}{\partial r}(E^2\psi) \\ - \frac{v_\theta\cos\theta}{r^2\sin^2\theta}(E^2\psi) + \frac{v_\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}(E^2\psi) \\ = \frac{\nu}{r\sin\theta}(E^4\psi) + \frac{v_r}{r^2\sin\theta}(E^2\psi) + \frac{v_\theta\cos\theta}{r^2\sin^2\theta}(E^2\psi)\end{aligned}$$

Combine like-terms.

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial t}(E^2\psi) + \frac{v_r}{r\sin\theta}\frac{\partial}{\partial r}(E^2\psi) + \frac{v_\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}(E^2\psi) - \frac{2v_r}{r^2\sin\theta}(E^2\psi) - \frac{2v_\theta\cos\theta}{r^2\sin^2\theta}(E^2\psi) = \frac{\nu}{r\sin\theta}(E^4\psi)$$

Multiply both sides by $r\sin\theta$.

$$\frac{\partial}{\partial t}(E^2\psi) + v_r\frac{\partial}{\partial r}(E^2\psi) + \frac{v_\theta}{r}\frac{\partial}{\partial\theta}(E^2\psi) - \frac{2v_r}{r}(E^2\psi) - \frac{2v_\theta\cos\theta}{r\sin\theta}(E^2\psi) = \nu E^4\psi$$

Now eliminate v_r and v_θ.

$$\begin{aligned}\frac{\partial}{\partial t}(E^2\psi) + \left(-\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\frac{\partial}{\partial r}(E^2\psi) + \frac{1}{r}\left(\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\right)\frac{\partial}{\partial\theta}(E^2\psi) \\ - \frac{2}{r}\left(-\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)(E^2\psi) - \left(\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\right)\frac{2\cos\theta}{r\sin\theta}(E^2\psi) = \nu E^4\psi\end{aligned}$$

Simplify the left side.

$$\frac{\partial}{\partial t}(E^2\psi) + \frac{1}{r^2\sin\theta}\left[\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}(E^2\psi) - \frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}(E^2\psi)\right] - \frac{2E^2\psi}{r^2\sin^2\theta}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right) = \nu E^4\psi$$

Notice that the quantity in square brackets can be written as a Jacobian.

$$\frac{\partial}{\partial t}(E^2\psi) + \frac{1}{r^2\sin\theta}\begin{vmatrix}\frac{\partial\psi}{\partial r} & \frac{\partial\psi}{\partial\theta} \\ \frac{\partial}{\partial r}(E^2\psi) & \frac{\partial}{\partial\theta}(E^2\psi)\end{vmatrix} - \frac{2E^2\psi}{r^2\sin^2\theta}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right) = \nu E^4\psi$$

Therefore, for a spherical axisymmetric flow with no ϕ-component of velocity or azimuthal dependence, the equation for the stream function is

$$\boxed{\frac{\partial}{\partial t}(E^2\psi) + \frac{1}{r^2\sin\theta}\frac{\partial(\psi, E^2\psi)}{\partial(r,\theta)} - \frac{2E^2\psi}{r^2\sin^2\theta}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right) = \nu E^4\psi.}$$

This is equation (D) in Table 4.2-1 on page 123.

Note that $E^4\psi$ expands as follows.

$$\begin{aligned}
E^4\psi &= E^2(E^2\psi) \\
&= \left(\frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\psi}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) \\
&= \frac{\partial^2}{\partial r^2}\left(\frac{\partial^2\psi}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) \\
&\quad - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial^2\psi}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) \\
&\quad + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial^2\psi}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) \\
&= \frac{\partial}{\partial r}\left(\frac{\partial^3\psi}{\partial r^3} + \frac{2\cot\theta}{r^3}\frac{\partial\psi}{\partial\theta} - \frac{\cot\theta}{r^2}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{2}{r^3}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial^3\psi}{\partial r\partial\theta^2}\right) \\
&\quad - \frac{\cot\theta}{r^2}\left(\frac{\partial^3\psi}{\partial\theta\partial r^2} + \frac{\csc^2\theta}{r^2}\frac{\partial\psi}{\partial\theta} - \frac{\cot\theta}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial^3\psi}{\partial\theta^3}\right) \\
&\quad + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial^3\psi}{\partial\theta\partial r^2} + \frac{\csc^2\theta}{r^2}\frac{\partial\psi}{\partial\theta} - \frac{\cot\theta}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial^3\psi}{\partial\theta^3}\right) \\
&= \left(\frac{\partial^4\psi}{\partial r^4} - \frac{6\cot\theta}{r^4}\frac{\partial\psi}{\partial\theta} + \frac{2\cot\theta}{r^3}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{2\cot\theta}{r^3}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{\cot\theta}{r^2}\frac{\partial^3\psi}{\partial r^2\partial\theta}\right. \\
&\qquad \left. + \frac{6}{r^4}\frac{\partial^2\psi}{\partial\theta^2} - \frac{2}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2} - \frac{2}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2} + \frac{1}{r^2}\frac{\partial^4\psi}{\partial r^2\partial\theta^2}\right) \\
&\quad + \left(-\frac{\cot\theta}{r^2}\frac{\partial^3\psi}{\partial\theta\partial r^2} - \frac{\cot\theta\csc^2\theta}{r^4}\frac{\partial\psi}{\partial\theta} + \frac{\cot^2\theta}{r^4}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\cot\theta}{r^4}\frac{\partial^3\psi}{\partial\theta^3}\right) \\
&\quad + \frac{1}{r^2}\left(\frac{\partial^4\psi}{\partial\theta^2\partial r^2} - \frac{2\cot\theta\csc^2\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{\csc^2\theta}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\csc^2\theta}{r^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial^3\psi}{\partial\theta^3} + \frac{1}{r^2}\frac{\partial^4\psi}{\partial\theta^4}\right) \\
&= \left(\frac{\partial^4\psi}{\partial r^4} - \frac{6\cot\theta}{r^4}\frac{\partial\psi}{\partial\theta} + \frac{2\cot\theta}{r^3}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{2\cot\theta}{r^3}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{\cot\theta}{r^2}\frac{\partial^3\psi}{\partial r^2\partial\theta}\right. \\
&\qquad \left. + \frac{6}{r^4}\frac{\partial^2\psi}{\partial\theta^2} - \frac{2}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2} - \frac{2}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2} + \frac{1}{r^2}\frac{\partial^4\psi}{\partial r^2\partial\theta^2}\right) \\
&\quad + \left(-\frac{\cot\theta}{r^2}\frac{\partial^3\psi}{\partial\theta\partial r^2} - \frac{\cot\theta\csc^2\theta}{r^4}\frac{\partial\psi}{\partial\theta} + \frac{\cot^2\theta}{r^4}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\cot\theta}{r^4}\frac{\partial^3\psi}{\partial\theta^3}\right) \\
&\quad + \left(\frac{1}{r^2}\frac{\partial^4\psi}{\partial\theta^2\partial r^2} - \frac{2\cot\theta\csc^2\theta}{r^4}\frac{\partial\psi}{\partial\theta} + \frac{\csc^2\theta}{r^4}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\csc^2\theta}{r^4}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\cot\theta}{r^4}\frac{\partial^3\psi}{\partial\theta^3} + \frac{1}{r^4}\frac{\partial^4\psi}{\partial\theta^4}\right) \\
&= \frac{\partial^4\psi}{\partial r^4} + \frac{4\cot\theta}{r^3}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{2\cot\theta}{r^2}\frac{\partial^3\psi}{\partial r^2\partial\theta} - \frac{4}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2} + \frac{2}{r^2}\frac{\partial^4\psi}{\partial r^2\partial\theta^2} \\
&\quad - \left(\frac{6\cot\theta}{r^4} + \frac{3\cot\theta\csc^2\theta}{r^4}\right)\frac{\partial\psi}{\partial\theta} + \left(\frac{6}{r^4} + \frac{\cot^2\theta}{r^4} + \frac{2\csc^2\theta}{r^4}\right)\frac{\partial^2\psi}{\partial\theta^2} - \frac{2\cot\theta}{r^4}\frac{\partial^3\psi}{\partial\theta^3} + \frac{1}{r^4}\frac{\partial^4\psi}{\partial\theta^4} \\
&= \frac{\partial^4\psi}{\partial r^4} + \frac{4\cot\theta}{r^3}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{4}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2} - \frac{2\cot\theta}{r^2}\frac{\partial^3\psi}{\partial r^2\partial\theta} + \frac{2}{r^2}\frac{\partial^4\psi}{\partial r^2\partial\theta^2} \\
&\quad - \frac{3\cot\theta(2+\csc^2\theta)}{r^4}\frac{\partial\psi}{\partial\theta} + \frac{5+3\csc^2\theta}{r^4}\frac{\partial^2\psi}{\partial\theta^2} - \frac{2\cot\theta}{r^4}\frac{\partial^3\psi}{\partial\theta^3} + \frac{1}{r^4}\frac{\partial^4\psi}{\partial\theta^4}
\end{aligned}$$

Part (d)

Suppose a fluid flows in one direction and varies in a perpendicular direction. Setup a Cartesian coordinate system so that the x-axis lies in the direction of the flow and that the velocity varies with respect to y.

$$\mathbf{v} = v_x(y,t)\hat{\mathbf{x}}$$

Assuming that the fluid density ρ is constant, the equation of continuity simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

If the fluid viscosity μ is also constant, then the equation of motion simplifies to the Navier-Stokes equation. Take the curl of both sides of it and use the resulting vorticity equation (either form) instead.

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{v} \cdot \nabla \mathbf{w} = \nu \nabla^2 \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}$$

Similar to the Navier-Stokes equation, it is a vector equation. It actually represents three scalar equations, one for each variable in the chosen coordinate system. The vorticity is defined to be the curl of velocity.

$$\mathbf{w} = \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & 0 & 0 \end{vmatrix} = -\frac{\partial v_x}{\partial y}\hat{\mathbf{z}}$$

Expanding the continuity equation in Cartesian coordinates tells us nothing.

$$\underbrace{\frac{\partial v_x}{\partial x}}_{=0} + \underbrace{\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0$$

Substitute this expression for $\mathbf{w}$ into the vorticity equation.

$$\frac{\partial}{\partial t}\left(-\frac{\partial v_x}{\partial y}\hat{\mathbf{z}}\right) + \mathbf{v} \cdot \nabla \left(-\frac{\partial v_x}{\partial y}\hat{\mathbf{z}}\right) = \nu \nabla^2 \left(-\frac{\partial v_x}{\partial y}\hat{\mathbf{z}}\right) + \left(-\frac{\partial v_x}{\partial y}\hat{\mathbf{z}}\right) \cdot \nabla \mathbf{v}$$

Let

$$\psi(y,t) = -\frac{\partial v_x}{\partial y}$$

in order to reduce the order of the PDE by one.

$$\frac{\partial \psi}{\partial t}\hat{\mathbf{z}} + \mathbf{v} \cdot \nabla [\psi(y,t)\hat{\mathbf{z}}] = \nu \nabla^2 [\psi(y,t)\hat{\mathbf{z}}] + [\psi(y,t)\hat{\mathbf{z}}] \cdot \nabla \mathbf{v}$$

Expand the second operator on the left side using equation (R) on page 832.

$$\begin{aligned} \mathbf{v} \cdot \nabla [\psi(y,t)\hat{\mathbf{z}}] &= v_x \frac{\partial}{\partial x}\psi(y,t)\hat{\mathbf{z}} \\ &= \mathbf{0} \end{aligned}$$

Expand the first operator on the right side using equation (M) on page 832.

$$\begin{aligned} \nu \nabla^2 [\psi(y,t)\hat{\mathbf{z}}] &= \nu \left[\frac{\partial^2}{\partial x^2}\psi(y,t) + \frac{\partial^2}{\partial y^2}\psi(y,t) + \frac{\partial^2}{\partial z^2}\psi(y,t)\right]\hat{\mathbf{z}} \\ &= \nu \frac{\partial^2 \psi}{\partial y^2}\hat{\mathbf{z}} \end{aligned}$$

Expand the second operator on the right side using equation (P) on page 832.

$$\begin{aligned}[\psi(y,t)\hat{\mathbf{z}}] \cdot \nabla \mathbf{v} &= \psi(y,t)\frac{\partial}{\partial z}v_x(y,t)\hat{\mathbf{x}} \\ &= \mathbf{0}\end{aligned}$$

With these results, the vorticity equation becomes

$$\frac{\partial \psi}{\partial t}\hat{\mathbf{z}} = \nu \frac{\partial^2 \psi}{\partial y^2}\hat{\mathbf{z}}.$$

Dot both sides by $\hat{\mathbf{z}}$ to get a scalar equation.

$$\boxed{\frac{\partial \psi}{\partial t} = \nu \frac{\partial^2 \psi}{\partial y^2}}$$

ψ satisfies the well-known diffusion equation. If the flow is steady, then the time derivative vanishes, and a third-order ODE for the velocity results.

$$0 = \nu \frac{d^2 \psi}{dy^2} \quad \rightarrow \quad 0 = \nu \frac{d^2}{dy^2}\left(-\frac{dv_x}{dy}\right) \quad \rightarrow \quad 0 = \frac{d^3 v_x}{dy^3}$$

This can be solved by integrating both sides with respect to y three times.

$$v_x(y) = \frac{C_1}{2}y^2 + C_2 y + C_3$$

Three boundary conditions are needed to determine these three arbitrary constants. See Problem 4B.4 for examples.

Suppose a fluid flows in one direction and varies radially. Setup a cylindrical coordinate system so that the z-axis lies in the direction of the flow and that the velocity varies with respect to r.

$$\mathbf{v} = v_z(r,t)\hat{\mathbf{z}}$$

Assuming that the fluid density ρ is constant, the equation of continuity simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

If the fluid viscosity μ is also constant, then the equation of motion simplifies to the Navier-Stokes equation. Take the curl of both sides of it and use the resulting vorticity equation (either form) instead.

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{v}\cdot\nabla\mathbf{w} = \nu\nabla^2\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{v}$$

Similar to the Navier-Stokes equation, it is a vector equation. It actually represents three scalar equations, one for each variable in the chosen coordinate system. The vorticity is defined to be the curl of velocity. Its expansion in cylindrical coordinates is given in equations (G), (H), and (I) on page 834.

$$\mathbf{w} = \nabla\times\mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\hat{\mathbf{z}} = -\frac{\partial v_z}{\partial r}\hat{\boldsymbol{\theta}}$$

Expanding the continuity equation in cylindrical coordinates tells us nothing.

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=\,0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}}_{=\,0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=\,0} = 0$$

Substitute this expression for $\mathbf{w}$ into the vorticity equation.

$$\frac{\partial}{\partial t}\left(-\frac{\partial v_z}{\partial r}\hat{\boldsymbol{\theta}}\right) + \mathbf{v}\cdot\nabla\left(-\frac{\partial v_z}{\partial r}\hat{\boldsymbol{\theta}}\right) = \nu\nabla^2\left(-\frac{\partial v_z}{\partial r}\hat{\boldsymbol{\theta}}\right) + \left(-\frac{\partial v_z}{\partial r}\hat{\boldsymbol{\theta}}\right)\cdot\nabla\mathbf{v}$$

Let

$$\psi(r,t) = -\frac{\partial v_z}{\partial r}$$

in order to reduce the order of the PDE by one.

$$\frac{\partial\psi}{\partial t}\hat{\boldsymbol{\theta}} + \mathbf{v}\cdot\nabla\left[\psi(r,t)\hat{\boldsymbol{\theta}}\right] = \nu\nabla^2\left[\psi(r,t)\hat{\boldsymbol{\theta}}\right] + \left[\psi(r,t)\hat{\boldsymbol{\theta}}\right]\cdot\nabla\mathbf{v}$$

Expand the second operator on the left side using equation (Q) on page 834.

$$\begin{aligned}\mathbf{v}\cdot\nabla\left[\psi(r,t)\hat{\boldsymbol{\theta}}\right] &= v_z\frac{\partial}{\partial z}\psi(r,t)\hat{\boldsymbol{\theta}} \\ &= \mathbf{0}\end{aligned}$$

Expand the first operator on the right side using equation (N) on page 834.

$$\begin{aligned}\nu\nabla^2\left[\psi(r,t)\hat{\boldsymbol{\theta}}\right] &= \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rw_\theta)\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 w_\theta}{\partial\theta^2}}_{=\,0} + \underbrace{\frac{\partial^2 w_\theta}{\partial z^2}}_{=\,0} + \underbrace{\frac{2}{r^2}\frac{\partial w_r}{\partial\theta}}_{=\,0}\right]\hat{\boldsymbol{\theta}} \\ &= \nu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\psi)\right)\hat{\boldsymbol{\theta}}\end{aligned}$$

Expand the second operator on the right side using equation (R) on page 834.

$$\begin{aligned}
\left[\psi(r,t)\hat{\boldsymbol{\theta}}\right]\cdot\nabla\mathbf{v} &= w_\theta\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}\right)\hat{\mathbf{z}} \\
&= \psi(r,t)\left(\frac{1}{r}\frac{\partial}{\partial\theta}v_z(r,t)\right)\hat{\mathbf{z}} \\
&= \mathbf{0}
\end{aligned}$$

With these results, the vorticity equation becomes

$$\frac{\partial\psi}{\partial t}\hat{\boldsymbol{\theta}} = \nu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\psi)\right)\hat{\boldsymbol{\theta}}.$$

Dot both sides by $\hat{\boldsymbol{\theta}}$ to get a scalar equation.

$$\boxed{\frac{\partial\psi}{\partial t} = \nu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\psi)\right)}$$

If the flow is steady, then the time derivative vanishes, and a third-order ODE for the velocity results.

$$0 = \nu\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r\psi)\right) \quad\rightarrow\quad 0 = \nu\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(-r\frac{dv_z}{dr}\right)\right] \quad\rightarrow\quad 0 = \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right]$$

This can be solved by integrating both sides with respect to r three times.

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right] = 0$$

Integrate both sides with respect to r.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = C_1$$

Multiply both sides by r.

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = C_1 r$$

Integrate both sides with respect to r once more.

$$r\frac{dv_z}{dr} = \frac{C_1}{2}r^2 + C_2$$

Divide both sides by r.

$$\frac{dv_z}{dr} = \frac{C_1}{2}r + \frac{C_2}{r}$$

Integrate both sides with respect to r once more.

$$v_z(r) = \frac{C_1}{4}r^2 + C_2\ln r + C_3$$

Three boundary conditions are needed to determine these three arbitrary constants. See Problem 4B.4 for examples.