Problem 3B.12

Pressure distribution in incompressible fluids. Penelope is staring at a beaker filled with a liquid, which for all practical purposes can be considered as incompressible; let its density be ρ_0. She tells you she is trying to understand how the pressure in the liquid varies with depth. She has taken the origin of coordinates at the liquid–air interface, with the positive z-axis pointing away from the liquid. She says to you:

"If I simplify the equation of motion for an incompressible liquid at rest, I get $0 = -dp/dz - \rho_0 g$. I can solve this and get $p = p_{\text{atm}} - \rho_0 gz$. That seems reasonable—the pressure increases with increasing depth.

"But, on the other hand, the equation of state for any fluid is $p = p(\rho, T)$, and if the system is at constant temperature, this just simplifies to $p = p(\rho)$. And, since the fluid is incompressible, $p = p(\rho_0)$, and p must be a constant throughout the fluid! How can that be?"

Clearly Penelope needs help. Provide a useful explanation.

Solution

What Penelope is missing is the fact that $p = p(\rho, T)$ is the equation of state for a gas (a compressible fluid) and doesn't apply to the liquid in the beaker. p is a function of ρ because increasing the density of gas molecules, for example, will lead to a greater number of collisions with the wall on average. p is a function of T because increasing the temperature will make the molecules move faster, resulting in harder collisions with the wall. For an incompressible fluid at constant temperature, on the other hand, $p \neq p(\rho)$. No matter how much pressure one applies to the liquid, its density will not change because it's incompressible.

$$p = p_{\text{atm}} - \rho_0 gz$$

is the appropriate equation for the pressure distribution in the liquid. It basically says that the pressure at z is due to the weight of the liquid above that level.