Problem 3B.13

Flow of a fluid through a sudden contraction.

(a) An incompressible liquid flows through a sudden contraction from a pipe of diameter D_1 into a pipe of smaller diameter D_2. What does the Bernoulli equation predict for $\mathscr{P}_1 - \mathscr{P}_2$, the difference between the modified pressures upstream and downstream of the contraction? Does this result agree with experimental observations?

(b) Repeat the derivation for the isothermal horizontal flow of an ideal gas through a sudden contraction.

Solution

Bernoulli's equation is valid for steady flow of an inviscid fluid. For two points along a streamline in the flow, the equation relates the velocities, pressures, and heights.

$$\frac{1}{2}(v_2^2 - v_1^2) + \int_{p_1}^{p_2} \frac{1}{\rho}\, dp + g(h_2 - h_1) = 0$$

Part (a)

The liquid is assumed to be incompressible, so the density is independent of pressure.

$$\frac{1}{2}(v_2^2 - v_1^2) + \frac{1}{\rho}(p_2 - p_1) + g(h_2 - h_1) = 0$$

Combine the pressure and gravity terms.

$$\frac{1}{2}(v_2^2 - v_1^2) + \frac{p_2 - p_1 + \rho g h_2 - \rho g h_1}{\rho} = 0$$

Introduce the modified pressure function $\mathscr{P}_i = p_i + \rho g h_i$.

$$\frac{1}{2}(v_2^2 - v_1^2) + \frac{\mathscr{P}_2 - \mathscr{P}_1}{\rho} = 0$$

Multiply both sides by ρ and solve for $\mathscr{P}_1 - \mathscr{P}_2$.

$$\mathscr{P}_1 - \mathscr{P}_2 = \frac{\rho}{2}(v_2^2 - v_1^2)$$

The Bernoulli equation predicts that if $\mathscr{P}_1 > \mathscr{P}_2$, then the fluid velocity (v_2) will be higher in the contracted part of the pipe than that (v_1) in the pipe of normal diameter.

Part (b)

For flow that occurs horizontally, $h_2 - h_1 = 0$.

$$\frac{1}{2}(v_2^2 - v_1^2) + \int_{p_1}^{p_2} \frac{1}{\rho}\, dp = 0$$

The equation of state for an ideal gas is known to be $pV = nRT$. Write this in terms of the gas's molar mass M and mass m.

$$pV = \frac{m}{M} RT$$

Solve for the density ρ.

$$\rho = \frac{m}{V} = \frac{pM}{RT}$$

Substitute this formula into the integrand in Bernoulli's equation.

$$\frac{1}{2}(v_2^2 - v_1^2) + \int_{p_1}^{p_2} \frac{RT}{pM}\, dp = 0$$

Evaluate the integral knowing that since the flow is isothermal, T is a constant.

$$\frac{1}{2}(v_2^2 - v_1^2) + \frac{RT}{M} \ln p \Big|_{p_1}^{p_2} = 0$$

Insert the limits and solve for the ratio of pressures.

$$\frac{1}{2}(v_2^2 - v_1^2) + \frac{RT}{M} \ln \frac{p_2}{p_1} = 0$$

$$\frac{1}{2}(v_2^2 - v_1^2) - \frac{RT}{M} \ln \frac{p_1}{p_2} = 0$$

$$\ln \frac{p_1}{p_2} = \frac{M}{2RT}(v_2^2 - v_1^2)$$

Therefore,

$$\frac{p_1}{p_2} = \exp\left[\frac{M}{2RT}(v_2^2 - v_1^2)\right].$$

The Bernoulli equation predicts that if $p_1 > p_2$, then the fluid velocity (v_2) will be higher in the contracted part of the pipe than that (v_1) in the pipe of normal diameter.