Problem 3B.3

Laminar flow in a square duct.

(a) A straight duct extends in the z direction for a length L and has a square cross section, bordered by the lines $x = \pm B$ and $y = \pm B$. A colleague has told you that the velocity distribution is given by

$$v_z = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{4\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right]\left[1 - \left(\frac{y}{B}\right)^2\right] \tag{3B.3-1}$$

Since this colleague has occasionally given you wrong advice in the past, you feel obliged to check the result. Does it satisfy the relevant boundary conditions and the relevant differential equation?

(b) According to the review article by Berker,[3] the mass rate of flow in a square duct is given by

$$w = \frac{0.563(\mathscr{P}_0 - \mathscr{P}_L)B^4\rho}{\mu L} \tag{3B.3-2}$$

Compare the coefficient in this expression with the coefficient that one obtains from Eq. 3B.3-1.

Solution

Part (a)

We assume that the fluid flows only in the z-direction and that the velocity varies as a function of x and y.

$$\mathbf{v} = v_z(x, y)\hat{\boldsymbol{z}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $x = \pm B$ and $y = \pm B$.

Boundary Condition 1:	$v_z(-B, y) = 0$
Boundary Condition 2:	$v_z(B, y) = 0$
Boundary Condition 3:	$v_z(x, -B) = 0$
Boundary Condition 4:	$v_z(x, B) = 0$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

[3] R. Berker, *Handbuch der Physik*, Vol. VIII/2, Springer, Berlin (1963); see pp. 67–77 for laminar flow in conduits of noncircular cross sections. See also W. E. Stewart, *AIChE Journal*, **8**, 425–428 (1962).

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using Cartesian coordinates is the appropriate choice for this problem, so equations (1) and (2) will be used in (x, y, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{\partial v_x}{\partial x}}_{=0} + \underbrace{\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_x}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_x}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_x}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_x}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial x} + \mu\left[\underbrace{\frac{\partial^2 v_x}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 v_x}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_x}{\partial z^2}}_{=0}\right] + \rho g_x$$

$$\rho\left(\underbrace{\frac{\partial v_y}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_y}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_y}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial y} + \mu\left[\underbrace{\frac{\partial^2 v_y}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 v_y}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_y}{\partial z^2}}_{=0}\right] + \rho g_y$$

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_z}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_z}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

The relevant equation for the velocity is the z-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_z(x, y)\hat{\boldsymbol{z}}$.

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right] + \rho g_z$$

The sum of $-\partial p/\partial z$ and ρg_z is the modified pressure gradient across the duct.

$$0 = -\frac{(\mathscr{P}_L - \mathscr{P}_0)}{L - 0} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right]$$

The velocity distribution thus satisfies the following PDE.

$$\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} = \frac{(\mathscr{P}_L - \mathscr{P}_0)}{\mu L}$$

We want to check whether the colleague's solution,

$$v_z = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{4\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right]\left[1 - \left(\frac{y}{B}\right)^2\right],$$

satisfies it. Find the second derivatives of v_z with respect to x and y.

$$\frac{\partial v_z}{\partial x} = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{4\mu L}\left(-\frac{2x}{B^2}\right)\left[1 - \left(\frac{y}{B}\right)^2\right]$$

$$\frac{\partial^2 v_z}{\partial x^2} = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{4\mu L}\left(-\frac{2}{B^2}\right)\left[1 - \left(\frac{y}{B}\right)^2\right]$$

$$\frac{\partial v_z}{\partial y} = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{4\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right]\left(-\frac{2y}{B^2}\right)$$

$$\frac{\partial^2 v_z}{\partial y^2} = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{4\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right]\left(-\frac{2}{B^2}\right)$$

Adding them together, we have

$$\begin{aligned}\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{4\mu L}\left(-\frac{2}{B^2}\right)\left[1 - \left(\frac{x}{B}\right)^2 + 1 - \left(\frac{y}{B}\right)^2\right] \\ &= \frac{(\mathscr{P}_L - \mathscr{P}_0)}{\mu L}\left(\frac{1}{2}\right)\left[2 - \left(\frac{x}{B}\right)^2 - \left(\frac{y}{B}\right)^2\right] \\ &\neq \frac{(\mathscr{P}_L - \mathscr{P}_0)}{\mu L},\end{aligned}$$

so the colleague's solution does not satisfy the PDE. Plugging in $x = \pm B$ and $y = \pm B$, we find that it does satisfy the boundary conditions, though.

Part (b)

The volumetric flow rate is given by the integral of the velocity field over the cross-sectional area the fluid is flowing through.

$$\frac{dV}{dt} = \iint v_z \, dA$$

To get the mass flow rate, multiply both sides by the fluid density ρ.

$$\rho \frac{dV}{dt} = \rho \iint v_z(x, y) \, dx \, dy$$

Since ρ is assumed to be constant, it can be brought inside the derivative on the left side. Density times volume gives mass.

$$\begin{aligned}\frac{dm}{dt} &= \rho \iint v_z(x, y) \, dx \, dy \\ &= \rho \int_{-B}^{B}\int_{-B}^{B} \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{4\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right]\left[1 - \left(\frac{y}{B}\right)^2\right] dx \, dy \\ &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2\rho}{4\mu L}\left[\int_{-B}^{B}\left(1 - \frac{x^2}{B^2}\right) dx\right]\left[\int_{-B}^{B}\left(1 - \frac{y^2}{B^2}\right) dy\right] \\ &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2\rho}{4\mu L}\left(\frac{4B}{3}\right)\left(\frac{4B}{3}\right)\end{aligned}$$

Using the colleague's solution then, we obtain the following mass flow rate. (Let $w = dm/dt$.)

$$\begin{aligned}w &= \frac{4(\mathscr{P}_0 - \mathscr{P}_L)B^4\rho}{9\mu L} \\ &\approx \frac{0.444(\mathscr{P}_0 - \mathscr{P}_L)B^4\rho}{\mu L}.\end{aligned}$$

Calculate the percent difference of this coefficient compared to the one obtained by Berker.

$$\frac{0.444 - 0.563}{0.563} \times 100\% \approx -21\%$$

We find that the colleague's solution predicts a mass flow rate that is about 21% less than the one predicted by Berker.[1]

[1] In my humble opinion this is how much the colleague's next paycheck deserves to be cut.

Part (c)

The aim here is to solve the governing equation for the velocity distribution,

$$\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} = \frac{(\mathscr{P}_L - \mathscr{P}_0)}{\mu L}, \tag{3}$$

subject to the reformulated boundary conditions.

$$\begin{aligned} v_z(0, y) &= 0 \\ v_z(2B, y) &= 0 \\ v_z(x, 0) &= 0 \\ v_z(x, 2B) &= 0 \end{aligned}$$

The origin of the coordinate system has been translated to the lower left corner of the square duct so that the problem is easier to solve. Since equation (3) is linear and inhomogeneous with homogeneous boundary conditions, the method of eigenfunction expansion can be applied to solve for v_z. Consider the eigenvalue problem of the operator involving the spatial variables

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = \lambda\phi \tag{4}$$

with the same boundary conditions,

$$\begin{aligned} \phi(0, y) &= 0 \\ \phi(2B, y) &= 0 \\ \phi(x, 0) &= 0 \\ \phi(x, 2B) &= 0. \end{aligned}$$

Equation (4) is known as the Helmholtz equation; since it is linear and homogeneous, we can use the method of separation of variables to solve it. Assume a product solution of the form $\phi = X(x)Y(y)$ and substitute it into the equation

$$X''Y + XY'' = \lambda XY$$

and the boundary conditions.

$$\begin{aligned} \phi(0, y) &= 0 & &\rightarrow & X(0)Y(y) &= 0 & &\rightarrow & X(0) &= 0 \\ \phi(2B, y) &= 0 & &\rightarrow & X(2B)Y(y) &= 0 & &\rightarrow & X(2B) &= 0 \\ \phi(x, 0) &= 0 & &\rightarrow & X(x)Y(0) &= 0 & &\rightarrow & Y(0) &= 0 \\ \phi(x, 2B) &= 0 & &\rightarrow & X(x)Y(2B) &= 0 & &\rightarrow & Y(2B) &= 0 \end{aligned} .$$

Now separate variables in the PDE: divide both sides by XY and bring the functions of y to the right side.

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda \quad \rightarrow \quad \frac{X''}{X} = \lambda - \frac{Y''}{Y}$$

The only way a function of x can equal a function of y is if both are equal to a constant η.

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y} = \eta$$

The ODE in X will be solved first to find the values of η for which there are solutions. Suppose first that η is positive: $\eta = \alpha^2$. Then

$$\frac{X''}{X} = \alpha^2$$

Multiply both sides by X.

$$X'' = \alpha^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Apply the boundary conditions here to find C_1 and C_2.

$$X(0) = C_1 = 0$$
$$X(2B) = C_1 \cosh 2\alpha B + C_2 \sinh 2\alpha B = 0$$

Since $C_1 = 0$, the second equation reduces to $C_2 \sinh 2\alpha B = 0$. Hyperbolic sine is not oscillatory, so $C_2 = 0$. The trivial solution is obtained, which means there are no positive values for η. Secondly, suppose $\eta = 0$. Then

$$\frac{X''}{X} = 0$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is obtained by integrating both sides with respect to x twice.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$X(0) = C_4 = 0$$
$$X(2B) = 2C_3 B + C_4 = 0$$

Solving this system of equations yields $C_3 = 0$ and $C_4 = 0$, resulting in the trivial solution, so η is not zero. Thirdly, suppose η is negative: $\eta = -\beta^2$. Then

$$\frac{X''}{X} = -\beta^2$$

Multiply both sides by X.

$$X'' = -\beta^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_5 \cos \beta x + C_6 \sin \beta x$$

Apply the boundary conditions here to find C_5 and C_6.

$$X(0) = C_5 = 0$$
$$X(2B) = C_5 \cos 2\beta B + C_6 \sin 2\beta B = 0$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sin 2\beta B = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$.

$$\begin{aligned} \sin 2\beta B &= 0 \\ 2\beta B &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{2B}, \quad n = 1, 2, \ldots \end{aligned}$$

Consequently, the function for $X(x)$ is

$$X(x) = C_6 \sin \beta x \quad \rightarrow \quad X_n(x) = \sin \frac{n\pi x}{2B}, \quad n = 1, 2, \ldots.$$

Now we solve the related ODE for $Y(y)$.

$$\lambda - \frac{Y''}{Y} = -\beta^2$$

Solve this equation for Y''.

$$Y'' = (\beta^2 + \lambda)Y$$

If λ is zero or positive, then the general solution for Y is written in terms of hyperbolic sine and hyperbolic cosine, and the trivial solution will result as before for X. Suppose then that λ is negative: $\lambda = -\gamma^2$.

$$\begin{aligned} Y'' &= (\beta^2 - \gamma^2)Y \\ &= -(\gamma^2 - \beta^2)Y \end{aligned}$$

The general solution is written in terms of sine and cosine.

$$Y(y) = C_7 \cos \sqrt{\gamma^2 - \beta^2} y + C_8 \sin \sqrt{\gamma^2 - \beta^2} y$$

Apply the boundary conditions here to determine C_7 and C_8.

$$\begin{aligned} Y(0) &= C_7 = 0 \\ Y(2B) &= C_7 \cos 2\sqrt{\gamma^2 - \beta^2} B + C_8 \sin 2\sqrt{\gamma^2 - \beta^2} B = 0 \end{aligned}$$

The second equation reduces to $C_8 \sin 2\sqrt{\gamma^2 - \beta^2} B = 0$. To avoid getting the trivial solution, we insist that $C_8 \neq 0$.

$$\begin{aligned} \sin 2\sqrt{\gamma^2 - \beta^2} B &= 0 \\ 2\sqrt{\gamma^2 - \beta^2} B &= m\pi, \quad m = 1, 2, \ldots \\ \gamma^2 - \beta^2 &= \left(\frac{m\pi}{2B}\right)^2 \\ \gamma^2 &= \beta^2 + \left(\frac{m\pi}{2B}\right)^2 \\ \gamma_{mn} &= \sqrt{\frac{n^2\pi^2}{4B^2} + \frac{m^2\pi^2}{4B^2}} \\ &= \frac{\pi}{2B}\sqrt{n^2 + m^2} \end{aligned}$$

Consequently, the function for $Y(y)$ is

$$Y(y) = C_8 \sin \sqrt{\gamma^2 - \beta^2} y \quad \rightarrow \quad Y_m(y) = \sin \frac{m\pi y}{2B}, \quad m = 1, 2, \ldots.$$

Thus, the eigenvalues are

$$\lambda_{mn} = -\gamma_{mn}^2 = -\frac{\pi^2}{4B^2}(n^2 + m^2), \qquad \begin{matrix} m = 1, 2, \ldots \\ n = 1, 2, \ldots \end{matrix},$$

and the eigenfunctions associated with them are

$$\phi_{mn}(x, y) = \sin \frac{n\pi x}{2B} \sin \frac{m\pi y}{2B}, \qquad \begin{matrix} m = 1, 2, \ldots \\ n = 1, 2, \ldots \end{matrix}.$$

The eigenfunctions of the Helmholtz equation form a complete set, so the unknown function $v_z(x, y)$ can be expanded in terms of them.

$$v_z(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{2B} \sin \frac{m\pi y}{2B}$$

To determine the coefficients A_{mn}, substitute this formula for v_z into equation (3).

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{2B} \sin \frac{m\pi y}{2B} = \frac{(\mathscr{P}_L - \mathscr{P}_0)}{\mu L}$$

Provided that v_z and its first derivatives with respect to x and y are continuous, the series can be differentiated term-by-term.

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \sin \frac{n\pi x}{2B} \sin \frac{m\pi y}{2B} = \frac{(\mathscr{P}_L - \mathscr{P}_0)}{\mu L}$$

The operator in parentheses applied to the eigenfunction is just λ_{mn} times the eigenfunction.

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \lambda_{mn} \sin \frac{n\pi x}{2B} \sin \frac{m\pi y}{2B} = \frac{(\mathscr{P}_L - \mathscr{P}_0)}{\mu L}$$

The left side is technically a double series, but it can be thought of as a Fourier sine series in y.

$$\sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} A_{mn} \lambda_{mn} \sin \frac{n\pi x}{2B}\right] \sin \frac{m\pi y}{2B} = \frac{(\mathscr{P}_L - \mathscr{P}_0)}{\mu L}$$

To solve for the term in square brackets, multiply both sides by $\sin \frac{p\pi y}{2B}$, where p is an integer,

$$\sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} A_{mn} \lambda_{mn} \sin \frac{n\pi x}{2B}\right] \sin \frac{m\pi y}{2B} \sin \frac{p\pi y}{2B} = \frac{(\mathscr{P}_L - \mathscr{P}_0)}{\mu L} \sin \frac{p\pi y}{2B}$$

and then integrate both sides with respect to y from 0 to $2B$.

$$\int_0^{2B} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} A_{mn} \lambda_{mn} \sin \frac{n\pi x}{2B}\right] \sin \frac{m\pi y}{2B} \sin \frac{p\pi y}{2B}\, dy = \int_0^{2B} \frac{(\mathscr{P}_L - \mathscr{P}_0)}{\mu L} \sin \frac{p\pi y}{2B}\, dy$$

Bring the constants in front of the integrals.

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} A_{mn}\lambda_{mn}\sin\frac{n\pi x}{2B}\right]\int_0^{2B}\sin\frac{m\pi y}{2B}\sin\frac{p\pi y}{2B}\,dy = \frac{(\mathscr{P}_L-\mathscr{P}_0)}{\mu L}\int_0^{2B}\sin\frac{p\pi y}{2B}\,dy$$

Because the sine functions are orthogonal, the integral on the left side is zero for $m \neq p$. As a result, every term in the infinite series vanishes except for one: $m = p$.

$$\left[\sum_{n=1}^{\infty} A_{mn}\lambda_{mn}\sin\frac{n\pi x}{2B}\right]\int_0^{2B}\sin^2\frac{m\pi y}{2B}\,dy = \frac{(\mathscr{P}_L-\mathscr{P}_0)}{\mu L}\int_0^{2B}\sin\frac{m\pi y}{2B}\,dy$$

Evaluate the integrals.

$$\left[\sum_{n=1}^{\infty} A_{mn}\lambda_{mn}\sin\frac{n\pi x}{2B}\right]\cdot B = \frac{(\mathscr{P}_L-\mathscr{P}_0)}{\mu L}\cdot\frac{2B}{m\pi}[1-(-1)^m]$$

Divide both sides by B.

$$\sum_{n=1}^{\infty} A_{mn}\lambda_{mn}\sin\frac{n\pi x}{2B} = \frac{(\mathscr{P}_L-\mathscr{P}_0)}{\mu L}\cdot\frac{2}{m\pi}[1-(-1)^m]$$

Use a similar procedure to get $A_{mn}\lambda_{mn}$. Multiply both sides by $\sin\frac{q\pi x}{2B}$,where q is an integer,

$$\sum_{n=1}^{\infty} A_{mn}\lambda_{mn}\sin\frac{n\pi x}{2B}\sin\frac{q\pi x}{2B} = \frac{(\mathscr{P}_L-\mathscr{P}_0)}{\mu L}\cdot\frac{2}{m\pi}[1-(-1)^m]\sin\frac{q\pi x}{2B}$$

and then integrate both sides with respect to x from 0 to $2B$.

$$\int_0^{2B}\sum_{n=1}^{\infty} A_{mn}\lambda_{mn}\sin\frac{n\pi x}{2B}\sin\frac{q\pi x}{2B}\,dx = \int_0^{2B}\frac{(\mathscr{P}_L-\mathscr{P}_0)}{\mu L}\cdot\frac{2}{m\pi}[1-(-1)^m]\sin\frac{q\pi x}{2B}\,dx$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} A_{mn}\lambda_{mn}\int_0^{2B}\sin\frac{n\pi x}{2B}\sin\frac{q\pi x}{2B}\,dx = \frac{(\mathscr{P}_L-\mathscr{P}_0)}{\mu L}\cdot\frac{2}{m\pi}[1-(-1)^m]\int_0^{2B}\sin\frac{q\pi x}{2B}\,dx$$

As explained before, only one term in the infinite series doesn't vanish as a result of the integration: $n = q$.

$$A_{mn}\lambda_{mn}\int_0^{2B}\sin^2\frac{n\pi x}{2B}\,dx = \frac{(\mathscr{P}_L-\mathscr{P}_0)}{\mu L}\cdot\frac{2}{m\pi}[1-(-1)^m]\int_0^{2B}\sin\frac{n\pi x}{2B}\,dx$$

Evaluate the integrals.

$$A_{mn}\lambda_{mn}\cdot B = \frac{(\mathscr{P}_L-\mathscr{P}_0)}{\mu L}\cdot\frac{2}{m\pi}[1-(-1)^m]\cdot\frac{2B}{n\pi}[1-(-1)^n]$$

Hence, the coefficients are

$$A_{mn} = \frac{4(\mathscr{P}_L-\mathscr{P}_0)}{\pi^2\mu L}\cdot\frac{1}{mn\lambda_{mn}}[1-(-1)^m][1-(-1)^n].$$

The velocity distribution is now known.

$$v_z(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4(\mathscr{P}_L - \mathscr{P}_0)}{\pi^2 \mu L} \cdot \frac{1}{mn\lambda_{mn}}[1-(-1)^m][1-(-1)^n] \sin\frac{n\pi x}{2B}\sin\frac{m\pi y}{2B}$$

Notice that if m is even or n is even, then the summand is zero. The result can thus be simplified (that is, made to converge faster) by summing over the odd integers only. Let $m = 2k-1$ and $n = 2l-1$ in the double series. Then $1-(-1)^m = 1-(-1)^n = 2$.

$$\begin{aligned}
v_z(x,y) &= \sum_{2k-1=1}^{\infty}\sum_{2l-1=1}^{\infty} \frac{4(\mathscr{P}_L - \mathscr{P}_0)}{\pi^2 \mu L} \cdot \frac{1}{(2k-1)(2l-1)\lambda_{(2k-1)(2l-1)}}(2)(2)\sin\left[\frac{(2l-1)\pi x}{2B}\right]\sin\left[\frac{(2k-1)\pi y}{2B}\right] \\
&= \frac{16(\mathscr{P}_L - \mathscr{P}_0)}{\pi^2 \mu L}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{(2k-1)(2l-1)\lambda_{(2k-1)(2l-1)}}\sin\left[\frac{(2l-1)\pi x}{2B}\right]\sin\left[\frac{(2k-1)\pi y}{2B}\right] \\
&= \frac{16(\mathscr{P}_L - \mathscr{P}_0)}{\pi^2 \mu L}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{(2k-1)(2l-1)\left(-\frac{\pi^2}{4B^2}\right)[(2k-1)^2+(2l-1)^2]}\sin\left[\frac{(2l-1)\pi x}{2B}\right]\sin\left[\frac{(2k-1)\pi y}{2B}\right] \\
&= \frac{64(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\pi^4 \mu L}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{(2k-1)(2l-1)[(2k-1)^2+(2l-1)^2]}\sin\left[\frac{(2l-1)\pi x}{2B}\right]\sin\left[\frac{(2k-1)\pi y}{2B}\right]
\end{aligned}$$

In the colleague's coordinate system, $x \in [-B, B]$ and $y \in [-B, B]$, the velocity distribution is therefore

$$\begin{aligned}
v_z(x,y) = \frac{64(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\pi^4 \mu L}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} &\frac{1}{(2k-1)(2l-1)[(2k-1)^2+(2l-1)^2]} \\
&\times \sin\left[\frac{(2l-1)\pi(x+B)}{2B}\right]\sin\left[\frac{(2k-1)\pi(y+B)}{2B}\right].
\end{aligned}$$

Part (d)

Here the average velocity, maximum velocity, and mass flow rate for flow in a square duct will be calculated. To find the average velocity, integrate the velocity distribution over the cross-sectional area that the fluid flows through and then divide by that area.

$$\begin{aligned}
\langle v_z \rangle = \frac{1}{(2B)^2}\int_0^{2B}\int_0^{2B} \frac{64(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\pi^4 \mu L}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} &\frac{1}{(2k-1)(2l-1)[(2k-1)^2+(2l-1)^2]} \\
&\times \sin\left[\frac{(2l-1)\pi x}{2B}\right]\sin\left[\frac{(2k-1)\pi y}{2B}\right]dx\,dy
\end{aligned}$$

$$\begin{aligned}
= \frac{16(\mathscr{P}_0 - \mathscr{P}_L)}{\pi^4 \mu L}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} &\frac{1}{(2k-1)(2l-1)[(2k-1)^2+(2l-1)^2]} \\
&\times \left\{\int_0^{2B}\sin\left[\frac{(2l-1)\pi x}{2B}\right]dx\right\}\left\{\int_0^{2B}\sin\left[\frac{(2k-1)\pi y}{2B}\right]dy\right\}
\end{aligned}$$

$$\begin{aligned}\langle v_z \rangle &= \frac{16(\mathscr{P}_0 - \mathscr{P}_L)}{\pi^4 \mu L} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2k-1)(2l-1)[(2k-1)^2 + (2l-1)^2]} \left[\frac{4B}{\pi(2l-1)}\right] \left[\frac{4B}{\pi(2k-1)}\right] \\ &= \frac{256(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\pi^6 \mu L} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2k-1)^2(2l-1)^2[(2k-1)^2 + (2l-1)^2]}\end{aligned}$$

Taking the first 1,000 terms in each sum, the double series is approximately 0.527926652. Therefore, the average velocity is approximately

$$\boxed{\langle v_z \rangle \approx \frac{0.141(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}.}$$

The maximum velocity occurs at the point furthest from the walls: $v_{z,\max} = v_z(B, B)$.

$$\begin{aligned}v_{z,\max} &= \frac{64(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\pi^4 \mu L} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2k-1)(2l-1)[(2k-1)^2 + (2l-1)^2]} \sin\left[\frac{(2l-1)\pi}{2}\right] \sin\left[\frac{(2k-1)\pi}{2}\right] \\ &= \frac{64(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\pi^4 \mu L} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2k-1)(2l-1)[(2k-1)^2 + (2l-1)^2]} (-1)^{l+1}(-1)^{k+1}\end{aligned}$$

Taking the first 1,000 terms in each sum, the double series is approximately 0.448516222. Therefore, the maximum velocity is approximately

$$\boxed{v_{z,\max} \approx \frac{0.295(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}.}$$

The mass flow rate is

$$\begin{aligned}w &= \rho \int_0^{2B} \int_0^{2B} \frac{64(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\pi^4 \mu L} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2k-1)(2l-1)[(2k-1)^2 + (2l-1)^2]} \\ &\qquad \times \sin\left[\frac{(2l-1)\pi x}{2B}\right] \sin\left[\frac{(2k-1)\pi y}{2B}\right] dx\, dy \\ &= \frac{64(\mathscr{P}_0 - \mathscr{P}_L)B^2 \rho}{\pi^4 \mu L} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2k-1)(2l-1)[(2k-1)^2 + (2l-1)^2]} \left[\frac{4B}{\pi(2l-1)}\right] \left[\frac{4B}{\pi(2k-1)}\right] \\ &= \frac{1024(\mathscr{P}_0 - \mathscr{P}_L)B^4 \rho}{\pi^6 \mu L} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2k-1)^2(2l-1)^2[(2k-1)^2 + (2l-1)^2]}.\end{aligned}$$

Taking the first 1,000 terms in each sum, the double series is approximately 0.527926652. Therefore, the mass flow rate is approximately

$$\boxed{w \approx \frac{0.562(\mathscr{P}_0 - \mathscr{P}_L)B^4 \rho}{\mu L}.}$$