Problem 3B.6

Circulating axial flow in an annulus (Fig. 3B.6). A rod of radius κR moves upward with a constant velocity v_0 through a cylindrical container of inner radius R containing a Newtonian liquid. The liquid circulates in the cylinder, moving upward along the moving central rod and moving downward along the fixed container wall. Find the velocity distribution in the annular region, far from the end disturbances. Flows similar to this occur in the seals of some reciprocating machinery—for example, in the annular space between piston rings.

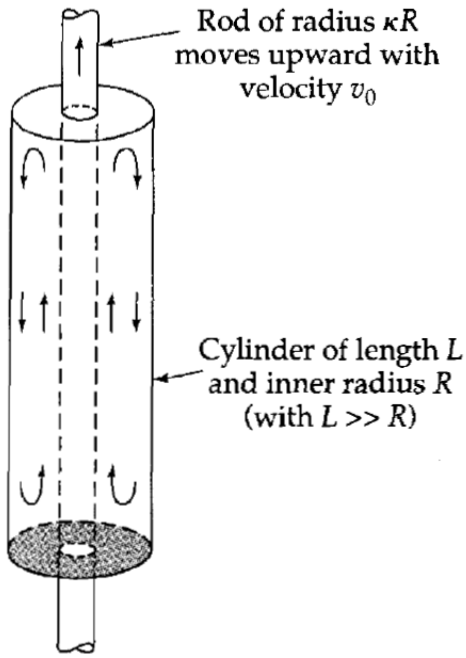

Fig. 3B.6. Circulating flow produced by an axially moving rod in a closed annular region.

(a) First consider the problem where the annular region is quite narrow—that is, where κ is just slightly less than unity. In that case the annulus may be approximated by a thin plane slit and the curvature can be neglected. Show that in this limit, the velocity distribution is given by

$$\frac{v_z}{v_0} = 3\left(\frac{\xi - \kappa}{1 - \kappa}\right)^2 - 4\left(\frac{\xi - \kappa}{1 - \kappa}\right) + 1 \tag{3B.6-1}$$

where $\xi = r/R$.

(b) Next work the problem without the thin-slit assumption. Show that the velocity distribution is given by

$$\frac{v_z}{v_0} = \frac{(1 - \xi^2)\left(1 - \dfrac{2\kappa^2}{1 - \kappa^2}\ln\dfrac{1}{\kappa}\right) - (1 - \kappa^2)\ln\dfrac{1}{\xi}}{(1 - \kappa^2) - (1 + \kappa^2)\ln\dfrac{1}{\kappa}} \tag{3B.6-2}$$

Solution

Part (b)

A cylindrical coordinate system will be used for this problem. We assume that the fluid flows only in the z-direction and that the velocity varies as a function of radius only.

$$\mathbf{v} = v_z(r)\hat{\mathbf{z}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $r = \kappa R$ and $r = R$.

$$\begin{aligned} &\text{Boundary Condition 1:} && v_z(\kappa R) = v_0 \\ &\text{Boundary Condition 2:} && v_z(R) = 0 \end{aligned}$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. From Appendix B.4 on page 846, the continuity equation in cylindrical coordinates is

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{v_\theta^2}{r}}_{=0}\right) = -\frac{\partial p}{\partial r} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_r}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_\theta}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

The relevant equation for the velocity is the z-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_z(r)\hat{\mathbf{z}}$.

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right] + \rho g_z$$

From the r- and θ-equations, we see that the pressure is independent of r and θ: $p = p(z)$. The z-axis points upward and gravity points downward, so $g_z = -g$.

$$0 = -\frac{dp}{dz} + \mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right] + \rho(-g)$$

Bring dp/dz and ρg to the other side.

$$\mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right] = \frac{dp}{dz} + \rho g$$

The only way a function of r can be equal to a function of z is if both are equal to a constant. Let the top of the cylinder be $z = L$ and the bottom of the cylinder be $z = 0$. The pressures at these heights are $p(L)$ and $p(0)$, respectively.

$$= \frac{p(L) - p(0)}{L - 0} + \rho g$$
$$= \frac{p(L) + \rho g L - p(0) - \rho g 0}{L}$$

Introduce the modified pressure $\mathscr{P}(z) = p(z) + \rho g z$ to get

$$\mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right] = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}.$$

This governing equation for the velocity and its associated boundary conditions will now be nondimensionalized to make it easier to solve.

$$\frac{\mu}{R^2}\left[\frac{R}{r}R\frac{d}{dr}\left(\frac{r}{R}R\frac{dv_z}{dr}\right)\right] = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}$$

Introduce the dimensionless radius ξ.

$$\xi = \frac{r}{R} \quad \to \quad d\xi = \frac{dr}{R} \quad \to \quad \frac{d}{d\xi} = R\frac{d}{dr}$$

As a result, the equation becomes

$$\frac{\mu}{R^2}\left[\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dv_z}{d\xi}\right)\right] = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}.$$

Nondimensionalize the velocity now.

$$\frac{\mu v_0}{R^2}\left[\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d}{d\xi}\frac{v_z}{v_0}\right)\right] = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}$$

Let it be denoted as $\phi = v_z/v_0$.

$$\frac{\mu v_0}{R^2}\left[\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\phi}{d\xi}\right)\right] = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}$$

Multiply both sides by $R^2\xi/\mu v_0$.

$$\frac{d}{d\xi}\left(\xi\frac{d\phi}{d\xi}\right) = \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{\mu L v_0}\xi$$

Integrate both sides with respect to ξ.

$$\xi \frac{d\phi}{d\xi} = \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0}\xi^2 + C_1$$

Divide both sides by ξ.

$$\frac{d\phi}{d\xi} = \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0}\xi + \frac{C_1}{\xi}$$

Integrate both sides with respect to ξ once more.

$$\phi(\xi) = \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}\xi^2 + C_1 \ln \xi + C_2$$

Use the boundary conditions for v_z to obtain those for ϕ

$$v_z(r = \kappa R) = v_0 \quad \rightarrow \quad \frac{v_z}{v_0}\left(\frac{r}{R} = \kappa\right) = 1 \quad \rightarrow \quad \phi(\xi = \kappa) = 1$$
$$v_z(r = R) = 0 \quad \rightarrow \quad \frac{v_z}{v_0}\left(\frac{r}{R} = 1\right) = 0 \quad \rightarrow \quad \phi(\xi = 1) = 0$$

and use them to determine C_1 and C_2.

$$\phi(\kappa) = \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}\kappa^2 + C_1 \ln \kappa + C_2 = 1$$
$$\phi(1) = \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0} + C_2 = 0$$

Solving this system of equations yields

$$C_1 = \frac{\frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}(\kappa^2 - 1) - 1}{\ln \frac{1}{\kappa}} \quad \text{and} \quad C_2 = -\frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}.$$

So the dimensionless velocity becomes

$$\begin{aligned}
\phi(\xi) &= \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}\xi^2 + \frac{\frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}(\kappa^2 - 1) - 1}{\ln \frac{1}{\kappa}} \ln \xi - \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0} \\
&= \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}(\xi^2 - 1) + \frac{\frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}(\kappa^2 - 1) - 1}{\ln \frac{1}{\kappa}} \ln \xi \\
&= \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}(\xi^2 - 1) + \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}\frac{(\kappa^2 - 1)}{\ln \frac{1}{\kappa}} \ln \xi - \frac{1}{\ln \frac{1}{\kappa}} \ln \xi \\
&= \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}\left(\xi^2 - 1 + \frac{\kappa^2 - 1}{\ln \frac{1}{\kappa}} \ln \xi\right) - \frac{1}{\ln \frac{1}{\kappa}} \ln \xi. \qquad (1)
\end{aligned}$$

The final answer is not in terms of the coefficient of the parentheses, so what we have to do is come up with another equation in order to eliminate it. Because the space in the cylinder is closed, all the liquid that rises as a result of the central rod's motion must come down along the sides. For any cross section then, the volumetric flow rate is equal to zero.

$$\int v_z(r)\, dA = 0,$$

where dA is an area element whose face is perpendicular to the direction the fluid is flowing.

$$\int_{\kappa R}^{R} v_z(r)(2\pi r\, dr) = 0$$

Divide both sides by $2\pi v_0 R^2$.

$$\int_{\kappa R}^{R} \frac{v_z}{v_0}(r)\left(\frac{r}{R}\frac{dr}{R}\right) = 0$$

In terms of the dimensionless variables defined earlier, this equation becomes

$$\int_{\kappa}^{1} \phi(\xi)\xi\, d\xi = 0.$$

Substitute the function found for ϕ here.

$$\int_{\kappa}^{1} \left[\frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}\left(\xi^2 - 1 + \frac{\kappa^2 - 1}{\ln\frac{1}{\kappa}}\ln\xi\right) - \frac{1}{\ln\frac{1}{\kappa}}\ln\xi\right]\xi\, d\xi = 0$$

Distribute ξ and split up the integral into three.

$$\frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}\left[\int_{\kappa}^{1}(\xi^3 - \xi)\, d\xi + \frac{\kappa^2 - 1}{\ln\frac{1}{\kappa}}\int_{\kappa}^{1}\xi\ln\xi\, d\xi\right] - \frac{1}{\ln\frac{1}{\kappa}}\int_{\kappa}^{1}\xi\ln\xi\, d\xi = 0$$

Multiply both sides by $\ln(1/\kappa)$.

$$\frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0}\left[\ln\frac{1}{\kappa}\int_{\kappa}^{1}(\xi^3 - \xi)\, d\xi + (\kappa^2 - 1)\int_{\kappa}^{1}\xi\ln\xi\, d\xi\right] - \int_{\kappa}^{1}\xi\ln\xi\, d\xi = 0$$

So then

$$\begin{aligned}
\frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{4\mu L v_0} &= \frac{\displaystyle\int_{\kappa}^{1}\xi\ln\xi\, d\xi}{\displaystyle\ln\frac{1}{\kappa}\int_{\kappa}^{1}(\xi^3 - \xi)\, d\xi + (\kappa^2 - 1)\int_{\kappa}^{1}\xi\ln\xi\, d\xi} \\
&= \frac{-\frac{1}{4}(1 + 2\kappa^2\ln\kappa - \kappa^2)}{\left(\ln\frac{1}{\kappa}\right)\left[-\frac{1}{4}(1-\kappa^2)^2\right] + (\kappa^2 - 1)\left[-\frac{1}{4}(1 + 2\kappa^2\ln\kappa - \kappa^2)\right]} \\
&= \frac{1 + 2\kappa^2\ln\kappa - \kappa^2}{(1-\kappa^2)\left[(1-\kappa^2)\ln\dfrac{1}{\kappa} - 1 - 2\kappa^2\ln\kappa + \kappa^2\right]} \\
&= \frac{1 + \dfrac{2\kappa^2}{1-\kappa^2}\ln\kappa}{(1-\kappa^2)\ln\dfrac{1}{\kappa} - 1 - 2\kappa^2\ln\kappa + \kappa^2} \\
&= \frac{1 - \dfrac{2\kappa^2}{1-\kappa^2}\ln\dfrac{1}{\kappa}}{\ln\dfrac{1}{\kappa}\left(1 + \kappa^2 - \dfrac{1-\kappa^2}{\ln\frac{1}{\kappa}}\right)}.
\end{aligned}$$

Picking up where we left off, the dimensionless velocity in equation (1) becomes

$$\begin{aligned}
\phi(\xi) &= \frac{1 - \dfrac{2\kappa^2}{1-\kappa^2}\ln\dfrac{1}{\kappa}}{\ln\dfrac{1}{\kappa}\left(1+\kappa^2 - \dfrac{1-\kappa^2}{\ln\frac{1}{\kappa}}\right)}\left(\xi^2 - 1 + \frac{\kappa^2-1}{\ln\frac{1}{\kappa}}\ln\xi\right) - \frac{1}{\ln\frac{1}{\kappa}}\ln\xi \\
&= \frac{\left(1 - \dfrac{2\kappa^2}{1-\kappa^2}\ln\dfrac{1}{\kappa}\right)\left(\xi^2 - 1 + \dfrac{\kappa^2-1}{\ln\frac{1}{\kappa}}\ln\xi\right) - \left(1+\kappa^2 - \dfrac{1-\kappa^2}{\ln\frac{1}{\kappa}}\right)\ln\xi}{\ln\dfrac{1}{\kappa}\left(1+\kappa^2 - \dfrac{1-\kappa^2}{\ln\frac{1}{\kappa}}\right)} \\
&= \frac{\left(1 - \dfrac{2\kappa^2}{1-\kappa^2}\ln\dfrac{1}{\kappa}\right)(\xi^2-1) + \cancel{\dfrac{\kappa^2-1}{\ln\frac{1}{\kappa}}\ln\xi} + 2\kappa^2\ln\xi - \ln\xi - \kappa^2\ln\xi - \cancel{\dfrac{\kappa^2-1}{\ln\frac{1}{\kappa}}\ln\xi}}{(1+\kappa^2)\ln\dfrac{1}{\kappa} - (1-\kappa^2)} \\
&= \frac{\left(1 - \dfrac{2\kappa^2}{1-\kappa^2}\ln\dfrac{1}{\kappa}\right)(\xi^2-1) + \kappa^2\ln\xi - \ln\xi}{(1+\kappa^2)\ln\dfrac{1}{\kappa} - (1-\kappa^2)} \\
&= \frac{\left(1 - \dfrac{2\kappa^2}{1-\kappa^2}\ln\dfrac{1}{\kappa}\right)(\xi^2-1) - (1-\kappa^2)\ln\xi}{(1+\kappa^2)\ln\dfrac{1}{\kappa} - (1-\kappa^2)} \\
&= \frac{\left(1 - \dfrac{2\kappa^2}{1-\kappa^2}\ln\dfrac{1}{\kappa}\right)(\xi^2-1) + (1-\kappa^2)\ln\dfrac{1}{\xi}}{(1+\kappa^2)\ln\dfrac{1}{\kappa} - (1-\kappa^2)}.
\end{aligned}$$

Therefore, multiplying the numerator and denominator by -1, the velocity distribution is

$$\frac{v_z}{v_0} = \frac{(1-\xi^2)\left(1 - \dfrac{2\kappa^2}{1-\kappa^2}\ln\dfrac{1}{\kappa}\right) - (1-\kappa^2)\ln\dfrac{1}{\xi}}{(1-\kappa^2) - (1+\kappa^2)\ln\dfrac{1}{\kappa}}.$$

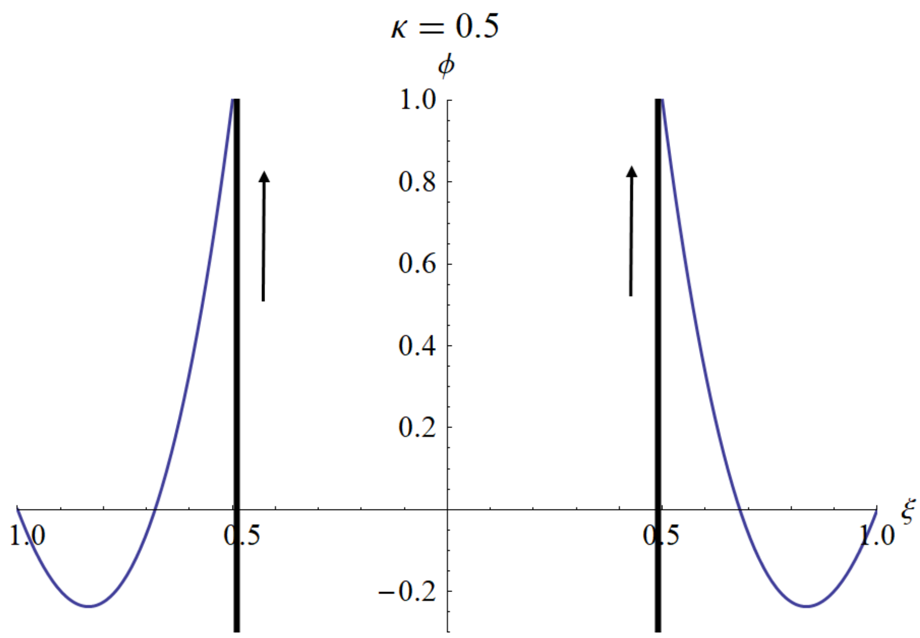

Figure 1: This is a sample plot of the dimensionless velocity versus ξ for $\kappa = 0.5$.

Part (a)

If the annular slit is narrow, then the problem can be solved in Cartesian coordinates to obtain an approximate solution. We assume that the fluid flows only in the z-direction and that the velocity varies as a function of x only.

$$\mathbf{v} = v_z(x)\hat{\mathbf{z}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $x = \kappa R$ and $x = R$.

$$\begin{aligned} &\text{Boundary Condition 1:} \qquad && v_z(\kappa R) = v_0 \\ &\text{Boundary Condition 2:} \qquad && v_z(R) = 0 \end{aligned}$$

From Appendix B.4 on page 846, the continuity equation in Cartesian coordinates is

$$\underbrace{\frac{\partial v_x}{\partial x}}_{=\,0} + \underbrace{\frac{\partial v_y}{\partial y}}_{=\,0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=\,0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in Cartesian coordinates.

$$\rho\bigg(\underbrace{\frac{\partial v_x}{\partial t}}_{=\,0} + \underbrace{v_x\frac{\partial v_x}{\partial x}}_{=\,0} + \underbrace{v_y\frac{\partial v_x}{\partial y}}_{=\,0} + \underbrace{v_z\frac{\partial v_x}{\partial z}}_{=\,0}\bigg) = -\frac{\partial p}{\partial x} + \mu\bigg[\underbrace{\frac{\partial^2 v_x}{\partial x^2}}_{=\,0} + \underbrace{\frac{\partial^2 v_x}{\partial y^2}}_{=\,0} + \underbrace{\frac{\partial^2 v_x}{\partial z^2}}_{=\,0}\bigg] + \underbrace{\rho g_x}_{=\,0}$$

$$\rho\bigg(\underbrace{\frac{\partial v_y}{\partial t}}_{=\,0} + \underbrace{v_x\frac{\partial v_y}{\partial x}}_{=\,0} + \underbrace{v_y\frac{\partial v_y}{\partial y}}_{=\,0} + \underbrace{v_z\frac{\partial v_y}{\partial z}}_{=\,0}\bigg) = -\frac{\partial p}{\partial y} + \mu\bigg[\underbrace{\frac{\partial^2 v_y}{\partial x^2}}_{=\,0} + \underbrace{\frac{\partial^2 v_y}{\partial y^2}}_{=\,0} + \underbrace{\frac{\partial^2 v_y}{\partial z^2}}_{=\,0}\bigg] + \underbrace{\rho g_y}_{=\,0}$$

$$\rho\bigg(\underbrace{\frac{\partial v_z}{\partial t}}_{=\,0} + \underbrace{v_x\frac{\partial v_z}{\partial x}}_{=\,0} + \underbrace{v_y\frac{\partial v_z}{\partial y}}_{=\,0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=\,0}\bigg) = -\frac{\partial p}{\partial z} + \mu\bigg[\frac{\partial^2 v_z}{\partial x^2} + \underbrace{\frac{\partial^2 v_z}{\partial y^2}}_{=\,0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=\,0}\bigg] + \rho g_z$$

The relevant equation for the velocity is the z-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_z(x)\hat{\mathbf{z}}$.

$$0 = -\frac{\partial p}{\partial z} + \mu \frac{d^2 v_z}{dx^2} + \rho g_z$$

From the x- and y-equations, we see that the pressure is independent of x and y: $p = p(z)$. The z-axis points upward and gravity points downward, so $g_z = -g$.

$$0 = -\frac{dp}{dz} + \mu \frac{d^2 v_z}{dx^2} + \rho(-g)$$

Bring dp/dz and ρg to the other side.

$$\mu \frac{d^2 v_z}{dx^2} = \frac{dp}{dz} + \rho g$$

The only way a function of x can be equal to a function of z is if both are equal to a constant. Let the top of the cylinder be $z = L$ and the bottom of the cylinder be $z = 0$. The pressures at these heights are $p(L)$ and $p(0)$, respectively.

$$\begin{aligned} &= \frac{p(L) - p(0)}{L - 0} + \rho g \\ &= \frac{p(L) + \rho g L - p(0) - \rho g 0}{L} \end{aligned}$$

Introduce the modified pressure $\mathscr{P}(z) = p(z) + \rho g z$ to get

$$\mu \frac{d^2 v_z}{dx^2} = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}.$$

This governing equation for the velocity and its associated boundary conditions will now be nondimensionalized to make it easier to solve.

$$\frac{\mu}{R^2}\left[R\frac{d}{dx}\left(R\frac{dv_z}{dx}\right)\right] = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}$$

Introduce the dimensionless coordinate ξ.

$$\xi = \frac{x}{R} \quad \rightarrow \quad d\xi = \frac{dx}{R} \quad \rightarrow \quad \frac{d}{d\xi} = R\frac{d}{dx}$$

As a result, the equation becomes

$$\frac{\mu}{R^2}\left[\frac{d}{d\xi}\left(\frac{dv_z}{d\xi}\right)\right] = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}.$$

Nondimensionalize the velocity now.

$$\frac{\mu v_0}{R^2}\left[\frac{d}{d\xi}\left(\frac{d}{d\xi}\frac{v_z}{v_0}\right)\right] = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}$$

Let it be denoted as $\phi = v_z/v_0$.

$$\frac{\mu v_0}{R^2}\left[\frac{d}{d\xi}\left(\frac{d\phi}{d\xi}\right)\right] = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}$$

Multiply both sides by $R^2/\mu v_0$.

$$\frac{d^2\phi}{d\xi^2} = \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{\mu L v_0}$$

Integrate both sides with respect to ξ.

$$\frac{d\phi}{d\xi} = \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{\mu L v_0}\xi + C_3$$

Integrate both sides with respect to ξ once more.

$$\phi(\xi) = \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0}\xi^2 + C_3\xi + C_4$$

Use the boundary conditions for v_z to obtain those for ϕ

$$v_z(x = \kappa R) = v_0 \quad \rightarrow \quad \frac{v_z}{v_0}\left(\frac{x}{R} = \kappa\right) = 1 \quad \rightarrow \quad \phi(\xi = \kappa) = 1$$
$$v_z(x = R) = 0 \quad \rightarrow \quad \frac{v_z}{v_0}\left(\frac{x}{R} = 1\right) = 0 \quad \rightarrow \quad \phi(\xi = 1) = 0$$

and use them to determine C_3 and C_4.

$$\phi(\kappa) = \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0}\kappa^2 + C_3\kappa + C_4 = 1$$
$$\phi(1) = \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0} + C_3 + C_4 = 0$$

Solving this system of equations yields

$$C_3 = -\frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0}(1+\kappa) - \frac{1}{1-\kappa} \quad \text{and} \quad C_4 = \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0}\kappa + \frac{1}{1-\kappa}.$$

So the dimensionless velocity becomes

$$\begin{aligned}\phi(\xi) &= \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0}\xi^2 - \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0}(1+\kappa)\xi - \frac{\xi}{1-\kappa} + \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0}\kappa + \frac{1}{1-\kappa} \\ &= \frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0}[\xi^2 - (1+\kappa)\xi + \kappa] + \frac{1}{1-\kappa}(1-\xi). \qquad (2)\end{aligned}$$

The final answer is not in terms of the coefficient of the square brackets, so what we have to do is come up with another equation in order to eliminate it. Because the space in the plane slit is closed, all the liquid that rises as a result of the central rod's motion must come down along the sides. For any cross section then, the volumetric flow rate is equal to zero.

$$\int v_z(x)\, dA = 0,$$

where dA is an area element whose face is perpendicular to the direction the fluid is flowing. Since curvature is neglected here, dA is the product of dx and Y, the length of the unfurled cylinder.

$$\int_{\kappa R}^{R} v_z(x)(Y\, dx) = 0$$

Divide both sides by Yv_0R.

$$\int_{\kappa R}^{R} \frac{v_z}{v_0}(x)\,\frac{dx}{R} = 0$$

In terms of the dimensionless variables defined earlier, this equation becomes

$$\int_{\kappa}^{1} \phi(\xi)\,d\xi = 0.$$

Substitute the function found for ϕ here.

$$\int_{\kappa}^{1} \left[\frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0}[\xi^2 - (1+\kappa)\xi + \kappa] + \frac{1}{1-\kappa}(1-\xi)\right] d\xi = 0$$

Split up the integral into two.

$$\frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0}\int_{\kappa}^{1}[\xi^2 - (1+\kappa)\xi + \kappa]\,d\xi + \frac{1}{1-\kappa}\int_{\kappa}^{1}(1-\xi)\,d\xi = 0$$

Solve for the factor we wish to eliminate.

$$\begin{aligned}
\frac{(\mathscr{P}_L - \mathscr{P}_0)R^2}{2\mu L v_0} &= \frac{\displaystyle\int_{\kappa}^{1}(\xi-1)\,d\xi}{(1-\kappa)\displaystyle\int_{\kappa}^{1}[\xi^2 - (1+\kappa)\xi + \kappa]\,d\xi} \\
&= \frac{-\frac{1}{2}(1-\kappa)^2}{(1-\kappa)\left[-\frac{1}{6}(1-\kappa)^3\right]} \\
&= \frac{3}{(1-\kappa)^2}
\end{aligned}$$

Picking up where we left off, the dimensionless velocity in equation (2) becomes

$$\begin{aligned}
\phi(\xi) &= \frac{3}{(1-\kappa)^2}[\xi^2 - (1+\kappa)\xi + \kappa] + \frac{1}{1-\kappa}(1-\xi) \\
&= \frac{3[\xi^2 - (1+\kappa)\xi + \kappa] + (1-\kappa)(1-\xi)}{(1-\kappa)^2} \\
&= \frac{3(\xi^2 - 2\kappa\xi + \kappa^2 + \kappa\xi + \kappa - \xi - \kappa^2) + (1-\kappa)(1-\xi)}{(1-\kappa)^2} \\
&= \frac{3(\xi^2 - 2\kappa\xi + \kappa^2)}{(1-\kappa)^2} + \frac{3(\kappa\xi + \kappa - \xi - \kappa^2) + (1-\kappa)(1-\xi)}{(1-\kappa)^2} \\
&= 3\frac{(\xi-\kappa)^2}{(1-\kappa)^2} + \frac{4\kappa\xi - 4\xi + 4\kappa - 4\kappa^2 + \kappa^2 - 2\kappa + 1}{(1-\kappa)^2} \\
&= 3\left(\frac{\xi-\kappa}{1-\kappa}\right)^2 + \frac{-4(\xi-\kappa)(1-\kappa)}{(1-\kappa)^2} + \frac{\kappa^2 - 2\kappa + 1}{(1-\kappa)^2}.
\end{aligned}$$

Therefore,

$$\frac{v_z}{v_0} = 3\left(\frac{\xi-\kappa}{1-\kappa}\right)^2 - 4\left(\frac{\xi-\kappa}{1-\kappa}\right) + 1.$$

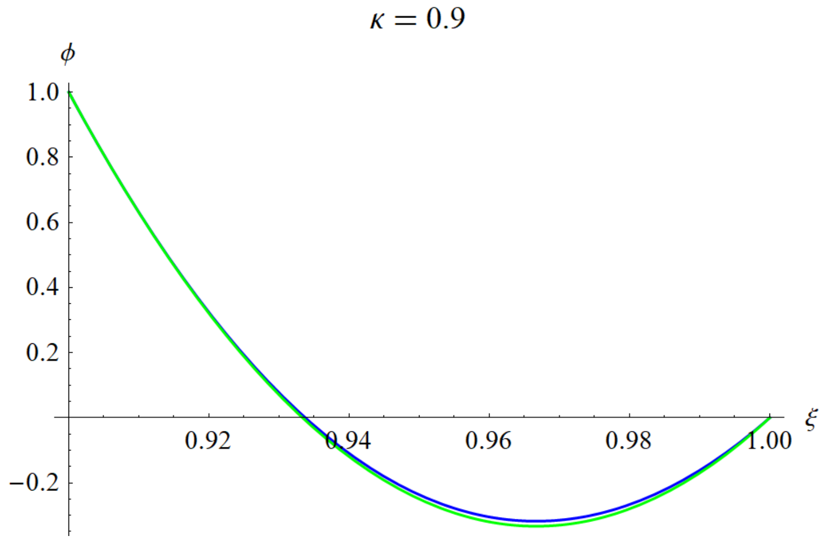

Figure 2: This is a side-by-side comparison of the approximate velocity distribution in the plane slit (in green) and the exact velocity distribution in the annulus (in blue) for $\kappa = 0.9$.

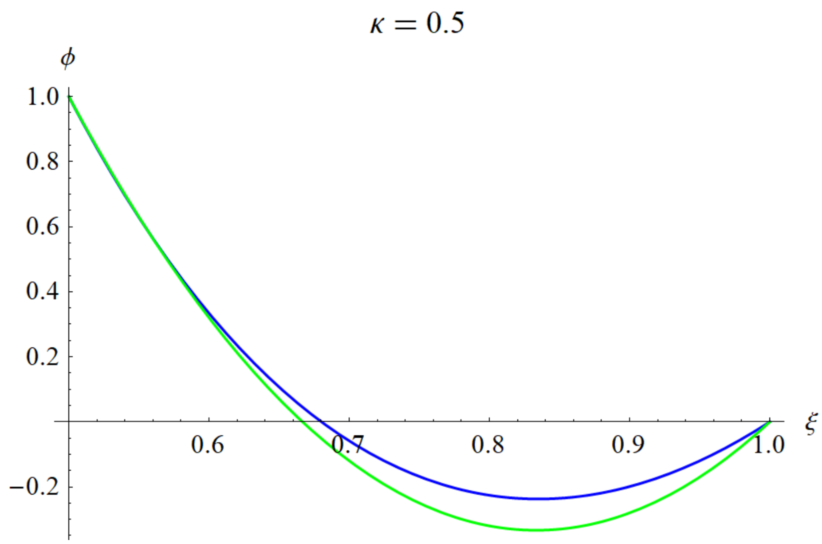

Figure 3: This is a side-by-side comparison of the approximate velocity distribution in the plane slit (in green) and the exact velocity distribution in the annulus (in blue) for $\kappa = 0.5$.