Problem 3C.3

Deformation of a fluid line (Fig. 3C.3). A fluid is contained in the annular space between two cylinders of radii κR and R. The inner cylinder is made to rotate with a constant angular velocity of Ω_i. Consider a line of fluid particles in the plane $z = 0$ extending from the inner cylinder to the outer cylinder and initially located at $\theta = 0$, normal to the two surfaces. How does this fluid line deform into a curve $\theta(r, t)$? What is the length, l, of the curve after N revolutions of the inner cylinder? Use Eq. 3.6-32.

$$\textit{Answer: } \frac{l}{R} = \int_{\kappa}^{1} \sqrt{1 + \frac{16\pi^2 N^2}{[(1/\kappa)^2 - 1]^2 \xi^4}}\, d\xi$$

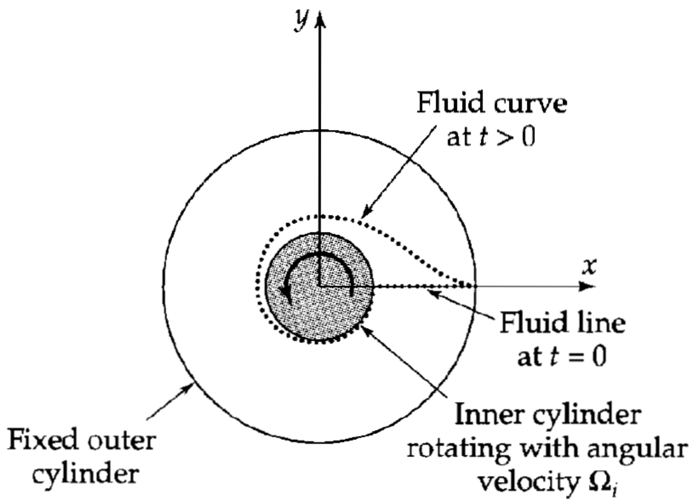

Fig. 3C.3. Deformation of a fluid line in Couette flow.

Solution

With the equation of motion the velocity distribution in the annular space can be determined. The key relationship between the curve θ and the (angular) velocity is

$$\frac{d\theta}{dt} = \omega$$
$$\frac{d\theta}{dt} = \frac{v}{r}.$$

Integrate both sides with respect to t,

$$\theta = \frac{v}{r}t + \theta_0,$$

assuming that the velocity is not a function of time. Since the line of fluid particles is initially located at $\theta = 0$, the constant of integration is zero.

$$\theta = \frac{v}{r}t \tag{1}$$

Once the velocity is determined, θ will be known and then we can find the length of the curve by calculating the arc length in polar coordinates. The fluid is assumed to flow only in the θ-direction and vary only in the r-direction.

$$\mathbf{v} = v_\theta(r)\hat{\boldsymbol{\theta}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $r = \kappa R$ and $r = R$.

Boundary Condition 1: $v_\theta(\kappa R) = \Omega_i \kappa R$

Boundary Condition 2: $v_\theta(R) = 0$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{2}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{3}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using cylindrical coordinates is the appropriate choice for this problem, so equations (2) and (3) will be used in (r, θ, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_r}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\right) = \underbrace{-\frac{1}{r}\frac{\partial p}{\partial \theta}}_{=0} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_\theta}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

The relevant equation for the velocity is the θ-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_\theta(r)\hat{\boldsymbol{\theta}}$.

$$0 = \mu\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right)$$

Divide both sides by μ.

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right) = 0$$

Integrate both sides with respect to r.

$$\frac{1}{r}\frac{d}{dr}(rv_\theta) = C_1$$

Multiply both sides by r.

$$\frac{d}{dr}(rv_\theta) = C_1 r$$

Integrate both sides with respect to r once more.

$$rv_\theta = C_1\frac{r^2}{2} + C_2$$

Divide both sides by r.

$$v_\theta(r) = C_1\frac{r}{2} + \frac{C_2}{r}$$

Apply the two boundary conditions here to determine C_1 and C_2.

$$\begin{aligned} v_\theta(\kappa R) &= C_1\frac{\kappa R}{2} + \frac{C_2}{\kappa R} = \Omega_i \kappa R \\ v_\theta(R) &= C_1\frac{R}{2} + \frac{C_2}{R} = 0 \end{aligned}$$

Solving the system of equations yields

$$C_1 = \frac{2(-\kappa^2\Omega_i)}{1-\kappa^2} \quad \text{and} \quad C_2 = \frac{\kappa^2 R^2(\Omega_i)}{1-\kappa^2}.$$

We then have for the velocity distribution

$$\begin{aligned} v_\theta(r) &= \frac{2(-\kappa^2\Omega_i)}{1-\kappa^2}\frac{r}{2} + \frac{\kappa^2 R^2(\Omega_i)}{1-\kappa^2}\frac{1}{r} \\ &= \frac{\kappa^2\Omega_i}{1-\kappa^2}\left(-r + \frac{R^2}{r}\right) \\ &= \frac{\Omega_i}{(1/\kappa)^2 - 1}\left(-r + \frac{R^2}{r}\right). \end{aligned}$$

From equation (1), then, the curve is

$$\theta(r,t) = \frac{\Omega_i t}{(1/\kappa)^2 - 1}\left(-1 + \frac{R^2}{r^2}\right).$$

Now the formula for arc length in polar coordinates will be calculated.

$$\begin{aligned} (ds)^2 &= (dx)^2 + (dy)^2 \\ &= \left[\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2\right](dr)^2 \end{aligned}$$

In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Apply the chain rule to find dx/dr and dy/dr.

$$\begin{aligned} &= \left[\left(\cos\theta - r\sin\theta\cdot\frac{\partial\theta}{\partial r}\right)^2 + \left(\sin\theta + r\cos\theta\cdot\frac{\partial\theta}{\partial r}\right)^2\right](dr)^2 \\ &= \left[\cos^2\theta - \cancel{2r\frac{\partial\theta}{\partial r}\sin\theta\cos\theta} + r^2\left(\frac{\partial\theta}{\partial r}\right)^2\sin^2\theta + \sin^2\theta + \cancel{2r\frac{\partial\theta}{\partial r}\sin\theta\cos\theta} + r^2\left(\frac{\partial\theta}{\partial r}\right)^2\cos^2\theta\right](dr)^2 \end{aligned}$$

$$\begin{aligned}(ds)^2 &= \left[(\sin^2\theta + \cos^2\theta) + r^2\left(\frac{\partial\theta}{\partial r}\right)^2(\sin^2\theta + \cos^2\theta)\right](dr)^2 \\ &= \left[1 + r^2\left(\frac{\partial\theta}{\partial r}\right)^2\right](dr)^2 \\ &= \left\{1 + r^2\left[\frac{\Omega_i t}{(1/\kappa)^2 - 1}\left(-2\frac{R^2}{r^3}\right)\right]^2\right\}(dr)^2 \\ &= \left\{1 + r^2\left[\frac{4(\Omega_i t)^2}{[(1/\kappa)^2 - 1]^2}\left(\frac{R^4}{r^6}\right)\right]\right\}(dr)^2 \\ &= \left[1 + \frac{4(\Omega_i t)^2}{[(1/\kappa)^2 - 1]^2}\left(\frac{R^4}{r^4}\right)\right](dr)^2\end{aligned}$$

Take the square root of both sides.

$$ds = \sqrt{1 + \frac{4(\Omega_i t)^2}{[(1/\kappa)^2 - 1]^2}\left(\frac{R^4}{r^4}\right)}\,dr$$

Integrate both sides.

$$\int_0^l ds = \int_{\kappa R}^R \sqrt{1 + \frac{4(\Omega_i t)^2}{[(1/\kappa)^2 - 1]^2}\left(\frac{R^4}{r^4}\right)}\,dr$$

Evaluate the integral on the left side and make the change of variables,

$$\begin{aligned}\xi &= \frac{r}{R} \\ d\xi &= \frac{dr}{R} \quad \rightarrow \quad R\,d\xi = dr,\end{aligned}$$

in the integral on the right side.

$$l = \int_\kappa^1 \sqrt{1 + \frac{4(\Omega_i t)^2}{[(1/\kappa)^2 - 1]^2}\left(\frac{1}{\xi^4}\right)}(R\,d\xi)$$

Divide both sides by R.

$$\frac{l}{R} = \int_\kappa^1 \sqrt{1 + \frac{4(\Omega_i t)^2}{[(1/\kappa)^2 - 1]^2}\left(\frac{1}{\xi^4}\right)}\,d\xi$$

The final answer is supposed to be in terms of N (the number of revolutions), not Ω_i or t, so we need another equation relating them. $\Omega_i t$ is the angle the inner cylinder rotates in time t, so $\Omega_i t/(2\pi) = N$.

$$\frac{l}{R} = \int_\kappa^1 \sqrt{1 + \frac{4(2\pi N)^2}{[(1/\kappa)^2 - 1]^2}\left(\frac{1}{\xi^4}\right)}\,d\xi$$

Therefore, the length of the curve after N revolutions of the inner cylinder is

$$\frac{l}{R} = \int_\kappa^1 \sqrt{1 + \frac{16\pi^2 N^2}{[(1/\kappa)^2 - 1]^2\xi^4}}\,d\xi.$$