Problem 4A.5

Numerical demonstration of the von Kármán momentum balance.

(a) Evaluate the integrals in Eq. 4.4-13 numerically for the Blasius velocity profile given in Fig. 4.4-3.

(b) Use the results of (a) to determine the magnitude of the wall shear stress $\tau_{yx}|_{y=0}$.

(c) Calculate the total drag force, F_x, for a plate of width W and length L, wetted on both sides. Compare your result with that obtained in Eq. 4.4-30.

Answers: **(a)** $\int_0^\infty \rho v_x(v_e - v_x)\,dy = 0.664\sqrt{\rho\mu v_\infty^3 x}$

$\int_0^\infty \rho(v_e - v_x)\,dy = 1.73\sqrt{\rho\mu v_\infty x}$

Solution

Part (a)

Eq. 4.4-13 on page 136 is the fabled von Kármán momentum balance.

$$\mu\frac{\partial v_x}{\partial y}\bigg|_{y=0} = \frac{d}{dx}\int_0^\infty \rho v_x(v_e - v_x)\,dy + \frac{dv_e}{dx}\int_0^\infty \rho(v_e - v_x)\,dy \tag{4.4-13}$$

Note that v_x represents the velocity distribution inside the boundary layer, and v_e represents the velocity external to the boundary layer. For tangential laminar flow along a flat plate in particular, $v_e = v_\infty$ and $v_x/v_\infty = f'$, where

$$f = f(\eta) = f\left(y\sqrt{\frac{v_\infty}{\nu x}}\right)$$

satisfies the Blasius equation,

$$-ff'' = 2f''',$$

subject to the following boundary conditions.

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 0 \\ \lim_{\eta\to\infty} f'(\eta) &= 1 \end{aligned}$$

The two integrals in the von Kármán momentum balance can be written as

$$\begin{array}{ll} \displaystyle\int_0^\infty \rho v_x(v_\infty - v_x)\,dy & \displaystyle\int_0^\infty \rho(v_\infty - v_x)\,dy \\ \displaystyle\int_0^\infty \rho v_\infty v_x\left(1 - \frac{v_x}{v_\infty}\right)dy & \displaystyle\int_0^\infty \rho v_\infty\left(1 - \frac{v_x}{v_\infty}\right)dy, \end{array}$$

and the differential dy can be written in terms of $d\eta$ by

$$d\eta = dy\sqrt{\frac{v_\infty}{\nu x}} \quad\rightarrow\quad \sqrt{v_\infty}\,dy = \sqrt{\nu x}\,d\eta = \sqrt{\frac{\mu x}{\rho}}\,d\eta.$$

As a result,

$$\int_0^\infty \rho\sqrt{v_\infty}v_x\left(1-\frac{v_x}{v_\infty}\right)\sqrt{v_\infty}\,dy \qquad \int_0^\infty \rho\sqrt{v_\infty}\left(1-\frac{v_x}{v_\infty}\right)\sqrt{v_\infty}\,dy,$$

$$\int_0^\infty \rho\sqrt{v_\infty}v_x\left(1-\frac{v_x}{v_\infty}\right)\sqrt{\frac{\mu x}{\rho}}\,d\eta \qquad \int_0^\infty \rho\sqrt{v_\infty}\left(1-\frac{v_x}{v_\infty}\right)\sqrt{\frac{\mu x}{\rho}}\,d\eta$$

$$\sqrt{\rho\mu v_\infty x}\int_0^\infty v_x\left(1-\frac{v_x}{v_\infty}\right)d\eta \qquad \sqrt{\rho\mu v_\infty x}\int_0^\infty \left(1-\frac{v_x}{v_\infty}\right)d\eta$$

$$\sqrt{\rho\mu v_\infty^3 x}\int_0^\infty \frac{v_x}{v_\infty}\left(1-\frac{v_x}{v_\infty}\right)d\eta \qquad \sqrt{\rho\mu v_\infty x}\int_0^\infty (1-f')\,d\eta$$

$$\sqrt{\rho\mu v_\infty^3 x}\int_0^\infty f'(1-f')\,d\eta \qquad \sqrt{\rho\mu v_\infty x}\int_0^\infty (1-f')\,d\eta.$$

A plot of the numerical solution to the Blasius equation is shown in Fig. 4.4-3.

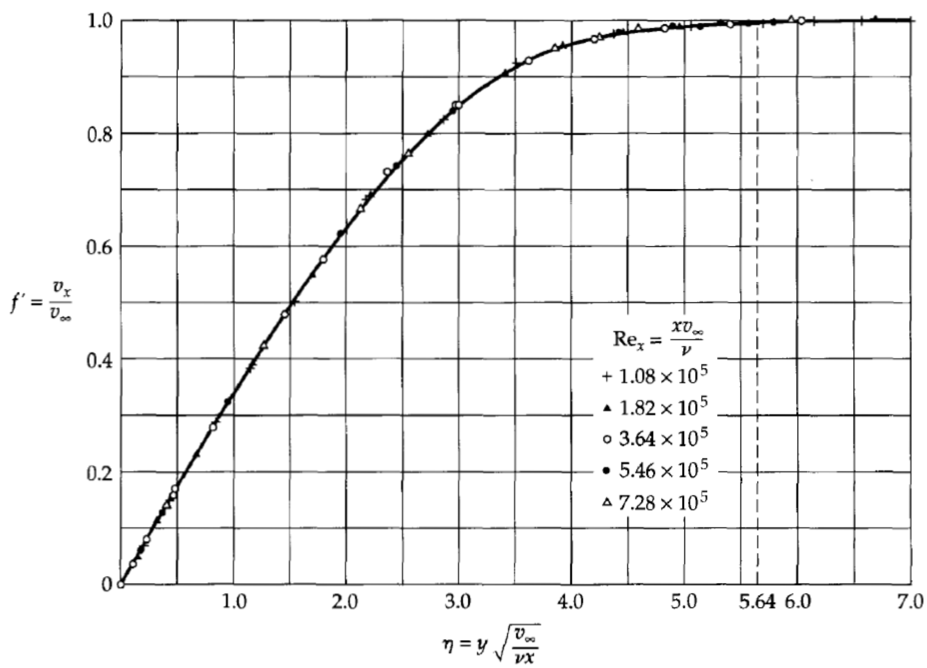

Fig. 4.4-3. Predicted and observed velocity profiles for tangential laminar flow along a flat plate. The solid line represents the solution of Eqs. **4.4-20 to 24**, obtained by Blasius [see H. Schlichting, *Boundary-Layer Theory*, McGraw-Hill, New York, 7th edition (1979), p. 137].

According to this graph, $f' \approx 1$ after $\eta = 7.0$, so $1 - f' \approx 0$ after $\eta = 7.0$. The two integrals can then be approximated by

$$\sqrt{\rho\mu v_\infty^3 x}\int_0^{7.0} f'(1-f')\,d\eta \qquad \sqrt{\rho\mu v_\infty x}\int_0^{7.0} (1-f')\,d\eta.$$

To evaluate these integrals numerically, apply Simpson's rule, which states that

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)],$$

where $\Delta x = (b-a)/n$ and n is even.

Therefore, taking $\Delta\eta$ to be 0.5,

$$\begin{aligned}
\int_0^\infty \rho v_x(v_e - v_x)\,dy &\approx \sqrt{\rho\mu v_\infty^3 x}\int_0^{7.0} f'(1-f')\,d\eta \\
&\approx \sqrt{\rho\mu v_\infty^3 x}\cdot\frac{\frac{7.0-0}{14}}{3}[0(1-0) + 4(0.17)(1-0.17) + 2(0.34)(1-0.34) + 4(0.49)(1-0.49) \\
&\quad + 2(0.63)(1-0.63) + 4(0.75)(1-0.75) + 2(0.85)(1-0.85) \\
&\quad + 4(0.92)(1-0.92) + 2(0.96)(1-0.96) + 4(0.98)(1-0.98) \\
&\quad + 2(0.99)(1-0.99) + 4(0.999)(1-0.999) + 2(0.9999)(1-0.9999) \\
&\quad + 4(0.99999)(1-0.99999) + 0.999999(1-0.999999)] \\
&\approx 0.6596\sqrt{\rho\mu v_\infty^3 x}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \rho(v_e - v_x)\,dy &\approx \sqrt{\rho\mu v_\infty x}\int_0^{7.0}(1-f')\,d\eta \\
&\approx \sqrt{\rho\mu v_\infty x}\cdot\frac{\frac{7.0-0}{14}}{3}[(1-0) + 4(1-0.17) + 2(1-0.34) + 4(1-0.49) \\
&\quad + 2(1-0.63) + 4(1-0.75) + 2(1-0.85) \\
&\quad + 4(1-0.92) + 2(1-0.96) + 4(1-0.98) \\
&\quad + 2(1-0.99) + 4(1-0.999) + 2(1-0.9999) \\
&\quad + 4(1-0.99999) + (1-0.999999)] \\
&\approx 1.704\sqrt{\rho\mu v_\infty x}.
\end{aligned}$$

These results are accurate to two significant figures. For better accuracy, use a table for the values of f' rather than trying to guess them from the graph in Fig. 4.4-3. Consult page 139 of the seventh edition of Schlichting's book, "Boundary Layer Theory," for example.

Part (b)

From Appendix B.1 on page 843, the shear stress τ_{yx} is

$$\tau_{yx} = -\mu\left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right],$$

but because the flow along a flat plate only moves horizontally in the x-direction, it reduces to

$$\tau_{yx} = -\mu\frac{\partial v_x}{\partial y}.$$

Take the absolute value of both sides and substitute this result into the left side of the von Kármán momentum balance.

$$\begin{aligned}
|\tau_{yx}|\big|_{y=0} &= \frac{d}{dx}\int_0^\infty \rho v_x(v_e - v_x)\,dy + \frac{dv_e}{dx}\int_0^\infty \rho(v_e - v_x)\,dy \\
&\approx \frac{d}{dx}(0.66\sqrt{\rho\mu v_\infty^3 x}) + \frac{dv_\infty}{dx}(1.7\sqrt{\rho\mu v_\infty x}) \\
&= 0.33\sqrt{\frac{\rho\mu v_\infty^3}{x}}
\end{aligned}$$

Part (c)

To obtain the total drag force, integrate the shear stress over the area of the flat plate. A factor of 2 is included since the plate is wetted on both sides.

$$\begin{aligned}
F_x &= 2\int |\tau_{yx}|\big|_{y=0}\, dA \\
&= 2\int_0^L |\tau_{yx}|\big|_{y=0}\,(W\,dx) \\
&= 2\int_0^L 0.33\sqrt{\frac{\rho\mu v_\infty^3}{x}}(W\,dx) \\
&= 2\times 0.33\sqrt{\rho\mu v_\infty^3}W\int_0^L \frac{dx}{\sqrt{x}} \\
&= 2\times 0.33\sqrt{\rho\mu v_\infty^3}W(2\sqrt{L}) \\
&\approx 1.3\sqrt{\rho\mu L W^2 v_\infty^3}
\end{aligned}$$

This is very close to the result in Eq. 4.4-30, which has a coefficient of 1.328. The percent difference is

$$\frac{1.3-1.328}{1.328}\times 100\% \approx -2.1\%.$$