Problem 4B.2

Flow near a wall suddenly set in motion (approximate solution) (Fig. 4B.2). Apply a procedure like that of Example 4.4-1 to get an approximate solution for Example 4.1-1.

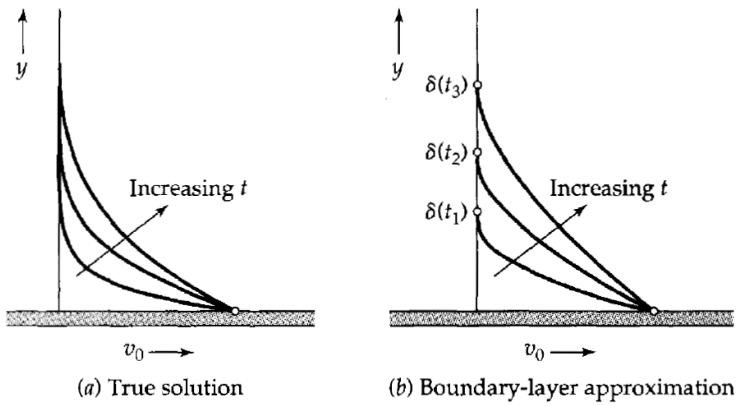


Fig. 4B.2. Comparison of true and approximate velocity profiles near a wall suddenly set in motion with velocity v_0.

(a) Integrate Eq. 4.1-1 over y to get

$$\int_0^\infty \frac{\partial v_x}{\partial t}\, dy = \nu \frac{\partial v_x}{\partial y}\bigg|_0^\infty \tag{4B.2-1}$$

Make use of the boundary conditions and the Leibniz rule for differentiating an integral (Eq. C.3-2) to rewrite Eq. 4B.2-1 in the form

$$\frac{d}{dt}\int_0^\infty \rho v_x\, dy = \tau_{yx}\big|_{y=0} \tag{4B.2-2}$$

Interpret this result physically.

(b) We know roughly what the velocity profiles look like. We can make the following reasonable postulate for the profiles:

$$\frac{v_x}{v_0} = 1 - \frac{3}{2}\frac{y}{\delta(t)} + \frac{1}{2}\left(\frac{y}{\delta(t)}\right)^3 \qquad \text{for } 0 \le y \le \delta(t) \tag{4B.2-3}$$

$$\frac{v_x}{v_0} = 1 \qquad \text{for } y \ge \delta(t) \tag{4B.2-4}$$

Here $\delta(t)$ is a time-dependent boundary-layer thickness. Insert this approximate expression into Eq. 4B.2-2 to obtain

$$\delta \frac{d\delta}{dt} = 4\nu \tag{4B.2-5}$$

(c) Integrate Eq. 4B.2-5 with a suitable initial value of $\delta(t)$, and insert the result into Eq. 4B.2-3 to get the approximate velocity profiles.

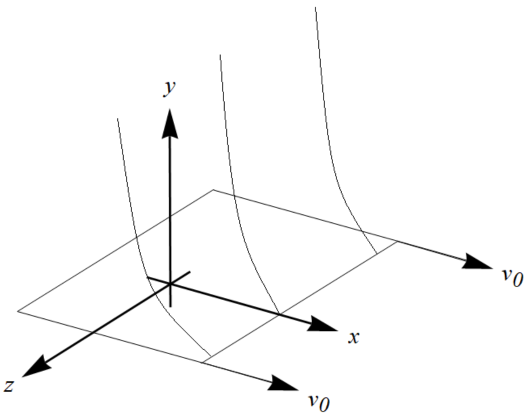
(d) Compare the values of v_x/v_0 obtained from (c) with those from Eq. 4.1-15 at $y/\sqrt{4\nu t} = 0.2,\ 0.5,$ and 1.0. Express the results as the ratio of the approximate value to the exact value.

Answer: **(d)** 1.015, 1.026, 0.738

TYPO: A colon is needed after "Answer."

Solution

The aim in this problem is to determine the velocity profile for a fluid that starts to move in response to a moving wall. This wall is coplanar with the $y = 0$ plane and moves in the x-direction with constant speed v_0 starting at $t = 0$.



The velocity is assumed to flow only in the x-direction and vary with y and t.

$$\mathbf{v} = v_x(y,t)\hat{\mathbf{x}}$$

Assuming the fluid does not slip on the wall, the velocity is v_0 at $y = 0$.

Boundary Condition 1: $v_x(0,t) = v_0, \quad t \geq 0$

Far, far away from the $y = 0$ plane, the velocity (and all its derivatives) will always be zero.

Boundary Condition 2: $\lim_{y\to\infty} v_x(y,t) = \lim_{y\to\infty} \frac{\partial v_x}{\partial y} = 0, \quad -\infty < t < \infty$

The fluid is initially at rest.

Initial Condition: $v_x(y,0) = 0, \quad 0 < y < \infty$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using Cartesian coordinates is the appropriate choice for this problem, so equations (1) and (2) will be expanded in (x, y, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{\partial v_x}{\partial x}}_{=0} + \underbrace{\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in Cartesian coordinates. (Gravity is assumed to be entirely in the y-direction, and no pressure gradients exist in the x- and z-directions.)

$$\rho\left(\frac{\partial v_x}{\partial t} + \underbrace{v_x\frac{\partial v_x}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_x}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_x}{\partial z}}_{=0}\right) = \underbrace{-\frac{\partial p}{\partial x}}_{=0} + \mu\left[\underbrace{\frac{\partial^2 v_x}{\partial x^2}}_{=0} + \frac{\partial^2 v_x}{\partial y^2} + \underbrace{\frac{\partial^2 v_x}{\partial z^2}}_{=0}\right] + \underbrace{\rho g_x}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_y}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_y}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_y}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial y} + \mu\left[\underbrace{\frac{\partial^2 v_y}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 v_y}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_y}{\partial z^2}}_{=0}\right] + \rho g_y$$

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_z}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_z}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = \underbrace{-\frac{\partial p}{\partial z}}_{=0} + \mu\left[\underbrace{\frac{\partial^2 v_z}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \underbrace{\rho g_z}_{=0}$$

The relevant equation for the velocity is the x-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_x(y, t)\hat{\boldsymbol{x}}$.

$$\rho\frac{\partial v_x}{\partial t} = \mu\frac{\partial^2 v_x}{\partial y^2}$$

Divide both sides by ρ and use the kinematic viscosity $\nu = \mu/\rho$.

$$\frac{\partial v_x}{\partial t} = \nu\frac{\partial^2 v_x}{\partial y^2}$$

This PDE was solved in Example 4.1-1 on page 135, and its solution is Eq. 4.1-15 on page 117.

$$v_x(y, t) = v_0 \operatorname{erfc}\frac{y}{\sqrt{4\nu t}} \tag{4.1-15}$$

Part (a)

Begin with the PDE for v_x.

$$\rho\frac{\partial v_x}{\partial t} = \mu\frac{\partial^2 v_x}{\partial y^2}$$

Integrate both sides with respect to y from 0 to ∞.

$$\int_0^\infty \rho\frac{\partial v_x}{\partial t}\,dy = \int_0^\infty \mu\frac{\partial^2 v_x}{\partial y^2}\,dy$$

Bring the constants in front.

$$\rho\int_0^\infty \frac{\partial v_x}{\partial t}\,dy = \mu\int_0^\infty \frac{\partial^2 v_x}{\partial y^2}\,dy$$

According to the Leibnitz rule,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)\,dx + \frac{db}{dt}\cdot f(b,t) - \frac{da}{dt}\cdot f(a,t).$$

Because the limits of integration in the previous equation are independent of t, the time derivative can be brought in front of the integral on the left side by the Leibnitz rule.

$$\rho\frac{d}{dt}\int_0^\infty v_x(y,t)\,dy = \mu\int_0^\infty \frac{\partial^2 v_x}{\partial y^2}\,dy$$

Bring ρ inside the derivative and the integral and evaluate the right side.

$$\begin{aligned}\frac{d}{dt}\int_0^\infty \rho v_x(y,t)\,dy &= \mu\int_0^\infty \frac{\partial^2 v_x}{\partial y^2}\,dy \\ &= \mu\frac{\partial v_x}{\partial y}\bigg|_0^\infty \\ &= \mu\bigg(\underbrace{\frac{\partial v_x}{\partial y}\bigg|_{y\to\infty}}_{=\,0} - \frac{\partial v_x}{\partial y}\bigg|_{y=0}\bigg) \\ &= -\mu\frac{\partial v_x}{\partial y}\bigg|_{y=0}\end{aligned}$$

According to Eq. B.1-4 on page 843, the shear stress τ_{yx} is

$$\tau_{yx} = -\mu\bigg[\underbrace{\frac{\partial v_y}{\partial x}}_{=\,0} + \frac{\partial v_x}{\partial y}\bigg] = -\mu\frac{\partial v_x}{\partial y}.$$

Therefore,

$$\frac{d}{dt}\int_0^\infty \rho v_x\,dy = \tau_{yx}\big|_{y=0}.$$

$\tau_{yx}\big|_{y=0}$ represents the shearing force per unit area in the x-direction that the wall in the $y = 0$ plane exerts on the fluid as it moves. $\tau_{yx}\big|_{y=0}$ depends on both the density and velocity at every height y above the wall. As amazing as this result is, it's nothing special; it's just another way to express Newton's second law as indicated by the derivation below.

$$\begin{aligned}F &= ma \\ F &= m\frac{dv}{dt} \\ F &= \frac{d(mv)}{dt} \\ F &= \frac{d(\rho V v)}{dt} \\ F &= \frac{d(\rho A y v)}{dt} \\ F &= A\frac{d(\rho y v)}{dt} \\ \frac{F}{A} &= \frac{d(\rho v y)}{dt}\end{aligned}$$

Part (b)

Begin with the PDE for v_x.

$$\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

Integrate both sides with respect to y from 0 to ∞.

$$\begin{aligned}
\int_0^\infty \rho \frac{\partial v_x}{\partial t}\, dy &= \int_0^\infty \mu \frac{\partial^2 v_x}{\partial y^2}\, dy \\
\rho \int_0^\infty \frac{\partial v_x}{\partial t}\, dy &= \mu \frac{\partial v_x}{\partial y}\bigg|_0^\infty \\
&= \mu \Bigg(\underbrace{\frac{\partial v_x}{\partial y}\bigg|_{y\to\infty}}_{=\,0} - \frac{\partial v_x}{\partial y}\bigg|_{y=0} \Bigg) \\
&= -\mu \frac{\partial v_x}{\partial y}\bigg|_{y=0}
\end{aligned}$$

Rather than solving the PDE for v_x, the velocity distribution will be approximated by something reasonable.

$$\begin{aligned}
v_x &= v_0 \left[1 - \frac{3}{2}\frac{y}{\delta(t)} + \frac{1}{2}\left(\frac{y}{\delta(t)}\right)^3 \right] && \text{for } 0 \le y \le \delta(t) \\
v_x &= v_0 && \text{for } y \ge \delta(t)
\end{aligned}$$

Substitute these formulas into the previous equation to get an ODE for $\delta(t)$.

$$\rho \left\{ \int_0^\delta \frac{\partial}{\partial t} v_0 \left[1 - \frac{3}{2}\frac{y}{\delta(t)} + \frac{1}{2}\left(\frac{y}{\delta(t)}\right)^3 \right] dy + \int_\delta^\infty \underbrace{\frac{\partial}{\partial t}(v_0)}_{=\,0}\, dy \right\} = -\mu \frac{\partial}{\partial y} \left\{ v_0 \left[1 - \frac{3}{2}\frac{y}{\delta(t)} + \frac{1}{2}\left(\frac{y}{\delta(t)}\right)^3 \right] \right\}\Bigg|_{y=0}$$

$$\begin{aligned}
\rho \left\{ \int_0^\delta v_0 \left[\frac{3}{2}\frac{y}{[\delta(t)]^2}\frac{d\delta}{dt} - \frac{3}{2} y^3 \frac{1}{[\delta(t)]^4}\frac{d\delta}{dt} \right] dy \right\} &= -\mu v_0 \left[-\frac{3}{2}\frac{1}{\delta(t)} + \frac{1}{2}\frac{3y^2}{[\delta(t)]^3} \right]\Bigg|_{y=0} \\
\rho v_0 \left[\frac{3}{2}\frac{1}{[\delta(t)]^2}\frac{d\delta}{dt}\int_0^\delta y\, dy - \frac{3}{2}\frac{1}{[\delta(t)]^4}\frac{d\delta}{dt}\int_0^\delta y^3\, dy \right] &= -\mu v_0 \left[-\frac{3}{2}\frac{1}{\delta(t)} \right] \\
\rho v_0 \left[\frac{3}{2}\frac{1}{[\delta(t)]^2}\frac{d\delta}{dt}\frac{[\delta(t)]^2}{2} - \frac{3}{2}\frac{1}{[\delta(t)]^4}\frac{d\delta}{dt}\frac{[\delta(t)]^4}{4} \right] &= \frac{3}{2}\mu v_0 \frac{1}{\delta(t)} \\
\rho v_0 \left(\frac{3}{4}\frac{d\delta}{dt} - \frac{3}{8}\frac{d\delta}{dt} \right) &= \frac{3}{2}\mu v_0 \frac{1}{\delta(t)} \\
\rho v_0 \left(\frac{3}{8}\frac{d\delta}{dt} \right) &= \frac{3}{2}\mu v_0 \frac{1}{\delta(t)} \\
\rho \frac{d\delta}{dt} &= 4\mu \frac{1}{\delta(t)}
\end{aligned}$$

Therefore, setting $\nu = \mu/\rho$,

$$\delta \frac{d\delta}{dt} = 4\nu.$$

Part (c)

Solve the ODE for δ.

$$\delta \frac{d\delta}{dt} = 4\nu$$

The left side can be rewritten using the chain rule.

$$\frac{1}{2}\frac{d}{dt}(\delta^2) = 4\nu$$

Multiply both sides by 2.

$$\frac{d}{dt}(\delta^2) = 8\nu$$

Integrate both sides with respect to t.

$$\delta^2 = 8\nu t + C_1$$

Assume that there is no boundary layer initially when the fluid is at rest: $\delta(0) = 0$.

$$0 = C_1$$

As a result, the previous equation becomes

$$\delta^2 = 8\nu t,$$

and the boundary layer thickness is

$$\delta(t) = \sqrt{8\nu t}.$$

Therefore, the postulated velocity distribution becomes

$$v_x = v_0\left[1 - \frac{3}{2}\frac{y}{\sqrt{8\nu t}} + \frac{1}{2}\left(\frac{y}{\sqrt{8\nu t}}\right)^3\right] \qquad \text{for } 0 \le y \le \sqrt{8\nu t}$$

$$v_x = v_0 \qquad \text{for } y \ge \sqrt{8\nu t}.$$

Part (d)

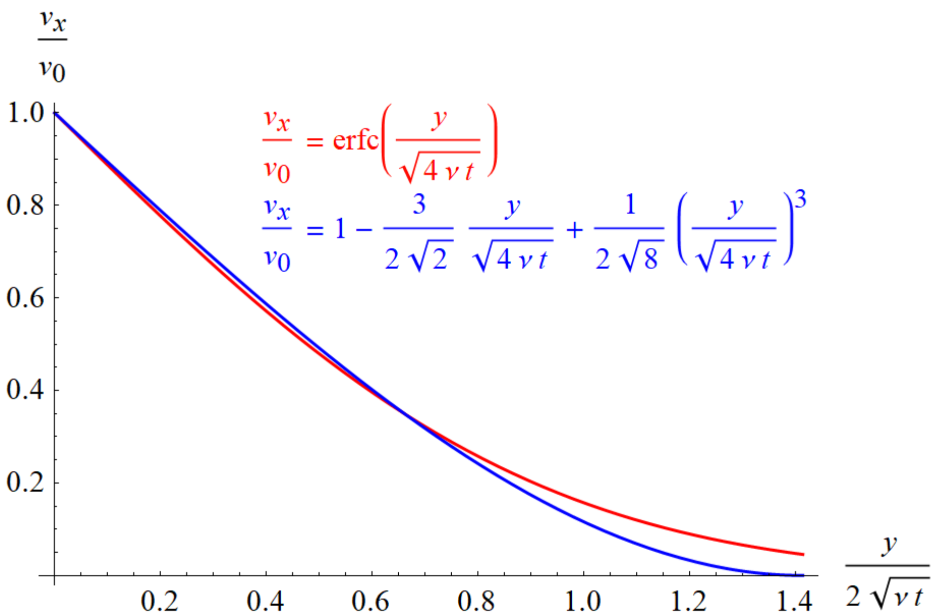
Write the velocity distribution in part (c) in terms of $y/\sqrt{4\nu t}$.

$$\frac{v_x}{v_0} = 1 - \frac{3}{2\sqrt{2}}\frac{y}{\sqrt{4\nu t}} + \frac{1}{2\sqrt{8}}\left(\frac{y}{\sqrt{4\nu t}}\right)^3 \qquad \text{for } 0 \le \frac{y}{\sqrt{4\nu t}} \le \sqrt{2}$$

$$\frac{v_x}{v_0} = 1 \qquad \text{for } \frac{y}{\sqrt{4\nu t}} \ge \sqrt{2}$$

For the values $y/\sqrt{4\nu t} = 0.2,\ 0.5,$ and 1.0, the first expression will be used because all three numbers are less than $\sqrt{2}$.

The graph below illustrates both the exact and approximate solutions for v_x for $0 \leq y/\sqrt{4\nu t} \leq \sqrt{2}$.



For the three values in particular, the ratios of the approximate solution to the exact solution are

$$\frac{y}{\sqrt{4\nu t}} = 0.2: \qquad \frac{\text{Approximate}}{\text{Exact}} = \frac{1 - \frac{3}{2\sqrt{2}}(0.2) + \frac{1}{2\sqrt{8}}(0.2)^3}{\operatorname{erfc}(0.2)} \approx 1.015$$

$$\frac{y}{\sqrt{4\nu t}} = 0.5: \qquad \frac{\text{Approximate}}{\text{Exact}} = \frac{1 - \frac{3}{2\sqrt{2}}(0.5) + \frac{1}{2\sqrt{8}}(0.5)^3}{\operatorname{erfc}(0.5)} \approx 1.026$$

$$\frac{y}{\sqrt{4\nu t}} = 1: \qquad \frac{\text{Approximate}}{\text{Exact}} = \frac{1 - \frac{3}{2\sqrt{2}}(1) + \frac{1}{2\sqrt{8}}(1)^3}{\operatorname{erfc}(1)} \approx 0.738.$$