# Problem 4B.3

**Creeping flow around a spherical bubble.** When a liquid flows around a gas bubble, circulation takes place within the bubble. This circulation lowers the interfacial shear stress, and, to a first approximation, we may assume that it is entirely eliminated. Repeat the development of Ex. 4.2-1 for such a gas bubble, assuming it is spherical.

**(a)** Show that B.C. 2 of Ex. 4.2-1 is replaced by

B.C. 2: at $r = R$, $$\frac{d}{dr}\left(\frac{1}{r^2}\frac{df}{dr}\right) + 2\frac{f}{r^4} = 0 \tag{4B.3-1}$$

and that the problem set-up is otherwise the same.

**(b)** Obtain the following velocity components:

$$v_r = v_\infty \left[1 - \left(\frac{R}{r}\right)\right] \cos\theta \tag{4B.3-2}$$

$$v_\theta = -v_\infty \left[1 - \frac{1}{2}\left(\frac{R}{r}\right)\right] \sin\theta \tag{4B.3-3}$$

**(c)** Next obtain the pressure distribution by using the equation of motion:

$$p = p_0 - \rho g h - \left(\frac{\mu v_\infty}{R}\right)\left(\frac{R}{r}\right)^2 \cos\theta \tag{4B.3-4}$$

**(d)** Evaluate the total force of the fluid on the sphere to obtain

$$F_z = \frac{4}{3}\pi R^3 \rho g + 4\pi\mu R v_\infty \tag{4B.3-5}$$

This result may be obtained by the method of §2.6 or by integrating the $z$-component of $-[\mathbf{n} \cdot \boldsymbol{\pi}]$ over the sphere surface ($\mathbf{n}$ being the outwardly directed unit vector normal to the surface of the sphere).

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**Solution**

## Part (a)

In order to model this gas bubble, consider a sphere that is stationary and immersed in a fluid that is flowing upward from the bottom. Use a spherical coordinate system with its origin at the sphere's center, where $r$ is the spherical coordinate, $\phi$ is the azimuthal angle, and $\theta$ is the polar angle.

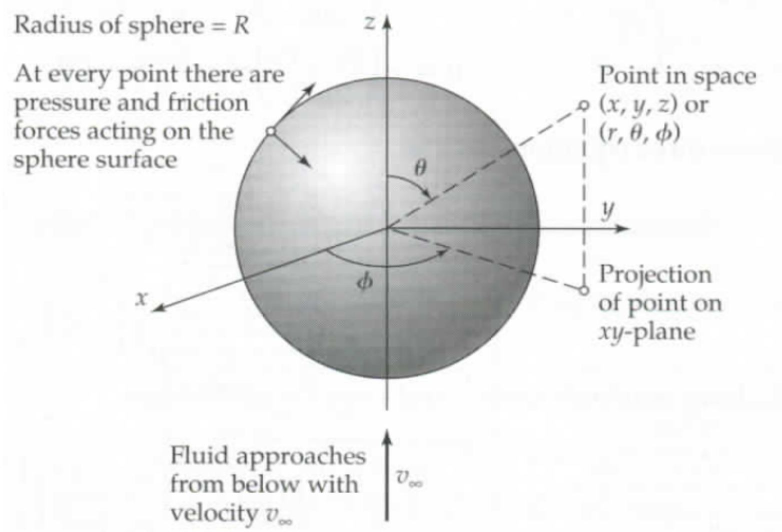



**Fig. 2.6-1** Sphere of radius $R$ around which a fluid is flowing. The coordinates $r$, $\theta$, and $\phi$ are shown. For more information on spherical coordinates, see Fig. A.8-2.

Because the flow is assumed to be steady and symmetric about the $z$-axis, the fluid velocity is independent of $\phi$ and $t$.

$$\mathbf{v} = v_r(r,\theta)\hat{\mathbf{r}} + v_\theta(r,\theta)\hat{\boldsymbol{\theta}}$$

Provided that the fluid density $\rho$ is constant, the equation of continuity simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

If the fluid viscosity $\mu$ is also constant, then the equation of motion simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

Taking the curl of both sides of the Navier-Stokes equation eliminates the pressure and gravity terms, resulting in the vorticity equation,

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{v} \cdot \nabla\mathbf{w} = \nu\nabla^2\mathbf{w} + \mathbf{w} \cdot \nabla\mathbf{v},$$

where $\mathbf{w} = \nabla \times \mathbf{v}$ is the vorticity and $\nu = \mu/\rho$ is the fluid's kinematic viscosity. For the particular case here where the flow is independent of $\phi$ and $t$, the continuity equation and vorticity equation may be combined by introducing a stream function $\psi = \psi(r,\theta)$ that is defined by

$$v_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}.$$

The equation that the stream function satisfies is given by

$$\underbrace{\frac{\partial}{\partial t}(E^2\psi)}_{=0} + \frac{1}{r^2\sin\theta}\left[\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}(E^2\psi) - \frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}(E^2\psi)\right] - \frac{2E^2\psi}{r^2\sin^2\theta}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right) = \nu E^4\psi,$$

where the operator $E^2$ is defined as

$$E^2\psi = \left(\frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi.$$

See part (c) of Problem 3D.2 for its derivation. With the creeping flow assumption, the nonlinear terms on the left side vanish.

$$0 = \nu E^4\psi$$

Divide both sides by $\nu$ and substitute the operator for $E^2$.

$$E^4\psi = 0$$

$$(E^2)(E^2)\psi = 0$$

$$\left(\frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi = 0$$

Associated with this PDE for $\psi$ are three boundary conditions. The first comes from the assumption that the bubble is impermeable: None of the fluid flowing around it can enter and none of the gas within it can leak out.

$$v_r(R,\theta) = 0 \quad \rightarrow \quad -\frac{1}{R^2\sin\theta}\frac{\partial\psi}{\partial\theta}\bigg|_{r=R} = 0 \quad \rightarrow \quad \frac{\partial\psi}{\partial\theta}\bigg|_{r=R} = 0$$

In Example 4.2-1 the second boundary condition came from the assumption that the fluid does not slip on the bubble's surface: $v_\theta(R,\theta) = 0$. Here, however, the assumption will be that the shear stress on the bubble acting in the $\theta$-direction is zero. Use the formula for $\tau_{r\theta}$ in spherical coordinates on page 844.

$$\begin{aligned} 0 &= (-\tau_{r\theta})|_{r=R} \\ &= \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\bigg|_{r=R} \end{aligned}$$

The third comes from the fact that far away from the sphere in any direction, the velocity is $\mathbf{v} = v_\infty\hat{\mathbf{z}}$. Use formula A.6-33 on page 828 to write this in terms of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$.

$$\begin{aligned} \mathbf{v} &= v_\infty[(\cos\theta)\hat{\mathbf{r}} + (-\sin\theta)\hat{\boldsymbol{\theta}}] \\ &= v_\infty\cos\theta\,\hat{\mathbf{r}} - v_\infty\sin\theta\,\hat{\boldsymbol{\theta}} \end{aligned}$$

This implies that

$$\begin{aligned} \lim_{r\to\infty} v_r(r,\theta) &= v_\infty\cos\theta \\ \lim_{r\to\infty} v_\theta(r,\theta) &= -v_\infty\sin\theta \end{aligned},$$

or in terms of the stream function,

$$\left.\begin{aligned} \lim_{r\to\infty} -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = v_\infty\cos\theta \quad &\rightarrow \quad \lim_{r\to\infty}\frac{\partial\psi}{\partial\theta} = -v_\infty r^2\sin\theta\cos\theta \\ \lim_{r\to\infty}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -v_\infty\sin\theta \quad &\rightarrow \quad \lim_{r\to\infty}\frac{\partial\psi}{\partial r} = -v_\infty r\sin^2\theta \end{aligned}\right\} \quad \Rightarrow \quad \lim_{r\to\infty}\psi(r,\theta) = -\frac{1}{2}v_\infty r^2\sin^2\theta.$$

Since the PDE for $\psi$ and all but one of its boundary conditions are linear and homogeneous, a separable solution is sought: $\psi(r,\theta) = f(r)\Theta(\theta)$. In particular, based on the form of the stream function in the third boundary condition, we hypothesize that the solution is of the form $\psi(r,\theta) = f(r)\sin^2\theta$. Then the second boundary condition becomes

$$\begin{aligned}
0 &= \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\bigg|_{r=R} \\
&= \mu\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\right)\right] + \frac{1}{r}\frac{\partial}{\partial\theta}\left(-\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right\}\bigg|_{r=R} \\
&= \mu\left\{r\frac{\partial}{\partial r}\left(\frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}[f(r)\sin^2\theta]\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}[f(r)\sin^2\theta]\right)\right\}\bigg|_{r=R} \\
&= \mu\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r^2\sin\theta}\left(\frac{df}{dr}\sin^2\theta\right)\right] + \frac{1}{r}\frac{\partial}{\partial\theta}\left[-\frac{1}{r^2\sin\theta}[2f(r)\sin\theta\cos\theta]\right]\right\}\bigg|_{r=R} \\
&= \mu\left[r\sin\theta\frac{d}{dr}\left(\frac{1}{r^2}\frac{df}{dr}\right) - \frac{2f(r)}{r^3}\frac{d}{d\theta}(\cos\theta)\right]\bigg|_{r=R} \\
&= \mu\left[r\sin\theta\frac{d}{dr}\left(\frac{1}{r^2}\frac{df}{dr}\right) + \frac{2f(r)}{r^3}\sin\theta\right]\bigg|_{r=R}.
\end{aligned}$$

Therefore, dividing both sides by $\mu r\sin\theta$, the second boundary condition is

$$\left[\frac{d}{dr}\left(\frac{1}{r^2}\frac{df}{dr}\right) + \frac{2f(r)}{r^4}\right]\bigg|_{r=R} = 0,$$

and the first boundary condition is

$$\begin{aligned}
0 &= \frac{\partial\psi}{\partial\theta}\bigg|_{r=R} \\
&= \frac{\partial}{\partial\theta}[f(r)\sin^2\theta]\bigg|_{r=R} \\
&= [2f(r)\sin\theta\cos\theta]|_{r=R} \\
&= 2f(R)\sin\theta\cos\theta \qquad \rightarrow \qquad f(R) = 0.
\end{aligned}$$

**Part (b)**

Now substitute the product solution $\psi(r,\theta) = f(r)\sin^2\theta$ into the PDE to obtain an ODE for $f$.

$$\begin{aligned}
0 &= E^4\psi \\
&= \left(\frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)[f(r)\sin^2\theta] \\
&= \left(\frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left[\frac{\partial^2}{\partial r^2}[f(r)\sin^2\theta] - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}[f(r)\sin^2\theta] + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}[f(r)\sin^2\theta]\right] \\
&= \left(\frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left[\frac{d^2f}{dr^2}\sin^2\theta - \frac{\cot\theta}{r^2}[2f(r)\sin\theta\cos\theta] + \frac{1}{r^2}f(r)(2\cos^2\theta - 2\sin^2\theta)\right] \\
&= \left(\frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left[\frac{d^2f}{dr^2}\sin^2\theta - \cancel{\frac{f(r)}{r^2}(2\cos^2\theta)} + \cancel{\frac{f(r)}{r^2}(2\cos^2\theta)} - \frac{f(r)}{r^2}(2\sin^2\theta)\right]
\end{aligned}$$

Continue simplifying the right side.

$$
\begin{aligned}
0 &= \left(\frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left[\frac{d^2 f}{dr^2}\sin^2\theta - \frac{2}{r^2}f(r)\sin^2\theta\right] \\
&= \frac{\partial^2}{\partial r^2}\left[\frac{d^2 f}{dr^2}\sin^2\theta - \frac{2}{r^2}f(r)\sin^2\theta\right] \\
&\quad - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\left[\frac{d^2 f}{dr^2}\sin^2\theta - \frac{2}{r^2}f(r)\sin^2\theta\right] \\
&\qquad + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left[\frac{d^2 f}{dr^2}\sin^2\theta - \frac{2}{r^2}f(r)\sin^2\theta\right] \\
&= \frac{d^4 f}{dr^4}\sin^2\theta - \frac{d^2}{dr^2}\left[\frac{2}{r^2}f(r)\right]\sin^2\theta \\
&\quad - \frac{\cot\theta}{r^2}\frac{d^2 f}{dr^2}(2\sin\theta\cos\theta) + \frac{2\cot\theta}{r^4}f(r)(2\sin\theta\cos\theta) \\
&\qquad + \frac{1}{r^2}\frac{d^2 f}{dr^2}(2\cos^2\theta - 2\sin^2\theta) - \frac{2}{r^4}f(r)(2\cos^2\theta - 2\sin^2\theta) \\
&= \frac{d^4 f}{dr^4}\sin^2\theta - \left(\frac{2}{r^2}\frac{d^2 f}{dr^2} - \frac{8}{r^3}\frac{df}{dr} + \frac{12}{r^4}f\right)\sin^2\theta \\
&\quad \cancel{- \frac{1}{r^2}\frac{d^2 f}{dr^2}(2\cos^2\theta)} + \cancel{\frac{2}{r^4}f(r)(2\cos^2\theta)} \\
&\qquad + \cancel{\frac{1}{r^2}\frac{d^2 f}{dr^2}(2\cos^2\theta)} - \frac{1}{r^2}\frac{d^2 f}{dr^2}(2\sin^2\theta) - \cancel{\frac{2}{r^4}f(r)(2\cos^2\theta)} + \frac{2}{r^4}f(r)(2\sin^2\theta) \\
&= \left(\frac{d^4 f}{dr^4} - \frac{4}{r^2}\frac{d^2 f}{dr^2} + \frac{8}{r^3}\frac{df}{dr} - \frac{8}{r^4}f\right)\sin^2\theta
\end{aligned}
$$

Divide both sides by $\sin^2\theta$ and then multiply both sides by $r^4$.

$$r^4\frac{d^4 f}{dr^4} - 4r^2\frac{d^2 f}{dr^2} + 8r\frac{df}{dr} - 8f = 0$$

This is a homogeneous equidimensional ODE, so its solutions are of the form $f = r^n$.

$$f = r^n \quad \rightarrow \quad \frac{df}{dr} = nr^{n-1} \quad \rightarrow \quad \frac{d^2 f}{dr^2} = n(n-1)r^{n-2} \quad \rightarrow \quad \frac{d^4 f}{dr^4} = n(n-1)(n-2)(n-3)r^{n-4}$$

Substitute these formulas into the ODE and solve for $n$.

$$r^4 n(n-1)(n-2)(n-3)r^{n-4} - 4r^2 n(n-1)r^{n-2} + 8rnr^{n-1} - 8r^n = 0$$

$$n(n-1)(n-2)(n-3)r^n - 4n(n-1)r^n + 8nr^n - 8r^n = 0$$

$$n(n-1)(n-2)(n-3) - 4n(n-1) + 8n - 8 = 0$$

$$n^4 - 6n^3 + 7n^2 + 6n - 8 = 0$$

$$(n+1)(n-1)(n-2)(n-4) = 0$$

$$n = \{-1, 1, 2, 4\}$$

Four solutions to the ODE are $f = r^{-1}$ and $f = r^1$ and $f = r^2$ and $f = r^4$. By the principle of superposition, the general solution is a linear combination of these four.

$$f(r) = C_1 r^{-1} + C_2 r + C_3 r^2 + C_4 r^4$$

The stream function is then

$$\psi(r,\theta) = \left(\frac{C_1}{r} + C_2 r + C_3 r^2 + C_4 r^4\right)\sin^2\theta.$$

From the third boundary condition,

$$\lim_{r\to\infty}\psi(r,\theta) = -\frac{1}{2}v_\infty r^2\sin^2\theta,$$

the stream function cannot be quartic in $r$ ($C_4 = 0$) and $C_3 = -v_\infty/2$. Apply the first two now to determine $C_1$ and $C_2$.

$$\begin{aligned} f(R) = 0 \qquad &\to \qquad \frac{C_1}{R} + C_2 R - \frac{v_\infty}{2}R^2 = 0 \\ \left[\frac{d}{dr}\left(\frac{1}{r^2}\frac{df}{dr}\right) + \frac{2f(r)}{r^4}\right]\bigg|_{r=R} = 0 \qquad &\to \qquad \frac{6C_1}{R^5} = 0 \end{aligned}$$

Solving this system of equations yields

$$C_1 = 0 \quad \text{and} \quad C_2 = \frac{v_\infty R}{2}.$$

Therefore, the stream function is

$$\psi(r,\theta) = \left(\frac{v_\infty R}{2}r - \frac{v_\infty}{2}r^2\right)\sin^2\theta,$$

and the resulting velocity components are

$$\begin{aligned} v_r(r,\theta) &= -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \\ &= -\frac{1}{r^2\sin\theta}\left(\frac{v_\infty R}{2}r - \frac{v_\infty}{2}r^2\right)(2\sin\theta\cos\theta) \\ &= \left(v_\infty - \frac{v_\infty R}{r}\right)\cos\theta \\ &= v_\infty\left[1 - \left(\frac{R}{r}\right)\right]\cos\theta \\ v_\theta(r,\theta) &= \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \\ &= \frac{1}{r\sin\theta}\left(\frac{v_\infty R}{2} - v_\infty r\right)\sin^2\theta \\ &= \left(\frac{v_\infty R}{2r} - v_\infty\right)\sin\theta \\ &= -v_\infty\left[1 - \frac{1}{2}\left(\frac{R}{r}\right)\right]\sin\theta. \end{aligned}$$

## Part (c)

To get the pressure distribution, return to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in spherical coordinates.

$$\begin{aligned}\rho\bigg(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}}_{=0} + \underbrace{\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi}}_{=0} - \underbrace{\frac{v_\theta^2+v_\phi^2}{r}}_{=0}\bigg) &= -\frac{\partial p}{\partial r} \\ &+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}}_{=0}\right] + \rho g_r\end{aligned}$$

$$\begin{aligned}\rho\bigg(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}}_{=0} + \underbrace{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}}_{=0} + \underbrace{\frac{v_rv_\theta - v_\phi^2\cot\theta}{r}}_{=0}\bigg) &= -\frac{1}{r}\frac{\partial p}{\partial\theta} \\ &+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right)\right. \\ &\left. + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2}}_{=0} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \underbrace{\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}}_{=0}\right] + \rho g_\theta\end{aligned}$$

$$\begin{aligned}\rho\bigg(\underbrace{\frac{\partial v_\phi}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\phi}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial\theta}}_{=0} + \underbrace{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}}_{=0} + \underbrace{\frac{v_\phi v_r + v_\theta v_\phi\cot\theta}{r}}_{=0}\bigg) &= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} \\ &+ \mu\left[\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta)\right)}_{=0}\right. \\ &\left. + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2}}_{=0} + \underbrace{\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}}_{=0} + \underbrace{\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi}}_{=0}\right] + \rho g_\phi\end{aligned}$$

Because the flow is steady and creeping, all the terms on the left side of each equation are zero. Gravity points down in the negative $z$-direction $\mathbf{g} = -g\hat{\mathbf{z}} = -g(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$, which means $g_r = -g\cos\theta$ and $g_\theta = g\sin\theta$ and $g_\phi = 0$.

$$\begin{aligned}0 &= -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right] - \rho g\cos\theta \\ 0 &= -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right] + \rho g\sin\theta \\ 0 &= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}\end{aligned}$$

Substitute the functions found for $v_r(r,\theta)$ and $v_\theta(r,\theta)$ and evaluate the derivatives.

$$\begin{aligned}
0 &= -\frac{\partial p}{\partial r} + \mu\left(\frac{2Rv_\infty \cos\theta}{r^3}\right) - \rho g \cos\theta \\
0 &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{Rv_\infty \sin\theta}{r^3}\right) + \rho g \sin\theta \\
0 &= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}
\end{aligned}$$

Solve for the pressure derivatives.

$$\begin{aligned}
\frac{\partial p}{\partial r} &= \frac{2\mu R v_\infty \cos\theta}{r^3} - \rho g \cos\theta \\
\frac{\partial p}{\partial \theta} &= \frac{\mu R v_\infty \sin\theta}{r^2} + \rho g r \sin\theta \\
\frac{\partial p}{\partial \phi} &= 0
\end{aligned}$$

This third equation implies that the pressure is independent of $\phi$: $p = p(r,\theta)$. Integrate both sides of the first equation partially with respect to $r$

$$p(r,\theta) = -\frac{\mu R v_\infty \cos\theta}{r^2} - \rho g r \cos\theta + F(\theta)$$

and then differentiate it with respect to $\theta$.

$$\frac{\partial p}{\partial \theta} = \frac{\mu R v_\infty \sin\theta}{r^2} + \rho g r \sin\theta + F'(\theta)$$

Comparing this with the second equation above, we see that

$$F'(\theta) = 0.$$

Integrate both sides with respect to $\theta$, setting the integration constant to $p_0$, the pressure at the $z = 0$ plane far away from the sphere.

$$F(\theta) = p_0$$

Therefore,

$$\begin{aligned}
p(r,\theta) &= -\frac{\mu R v_\infty \cos\theta}{r^2} - \rho g r \cos\theta + p_0 \\
&= p_0 - \rho g r \cos\theta - \frac{\mu v_\infty R \cos\theta}{r^2} \\
&= p_0 - \rho g h - \frac{\mu v_\infty}{R}\left(\frac{R}{r}\right)^2 \cos\theta,
\end{aligned}$$

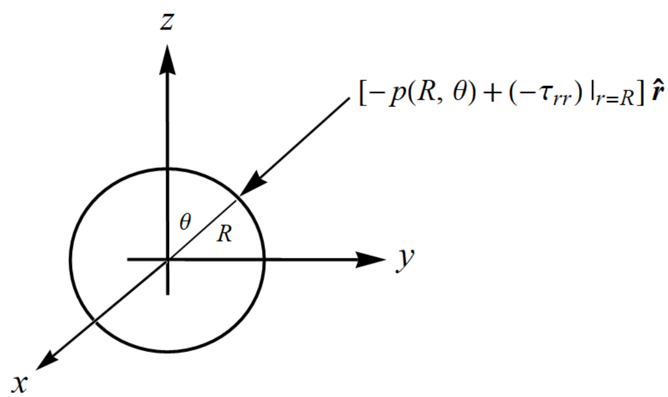
where $h = r\cos\theta$.

## Part (d)

The total force acting on the bubble is due to the pressure, the viscous shear stress, and the viscous normal stress. It can be split into two components, the first acting normally and the second acting tangentially.

$$\mathbf{F} = \mathbf{F}_\perp + \mathbf{F}_\parallel$$

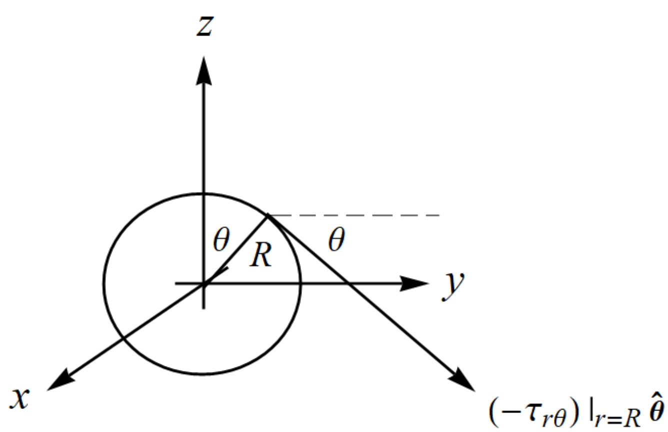
The force per unit area acting normally is due to the pressure and viscous normal stress as shown below.




Integrate it over the surface area of the sphere to get the force. A factor of $\cos\theta$ is needed in the integrand to get the $z$-component of force specifically. Note that $p$ has a minus sign in front of it because the force resulting from it acts radially inward. Also, $\tau_{rr}$ has a minus sign in front of it because the fluid is in a region of greater $r$ acting on a surface of lesser $r$. No second minus sign is needed in front of $\tau_{rr}$ because the velocity components that it's in terms of already take care of it.

$$\begin{aligned}
(F_\perp)_z &= \int [-p(R,\theta) + (-\tau_{rr})|_{r=R}]\hat{\mathbf{r}}(\cos\theta)\cdot d\mathbf{A} \\
&= \int \left[-\left(p_0 - \rho g R\cos\theta - \frac{\mu v_\infty}{R}\cos\theta\right) + 2\mu \left.\frac{\partial v_r}{\partial r}\right|_{r=R}\right]\hat{\mathbf{r}}(\cos\theta)\cdot(\hat{\mathbf{r}}\,dA) \\
&= \int \left(-p_0 + \rho g R\cos\theta + \frac{\mu v_\infty}{R}\cos\theta + \frac{2\mu v_\infty}{R}\cos\theta\right)(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})\cos\theta\,dA \\
&= \int \left(-p_0 + \rho g R\cos\theta + \frac{3\mu v_\infty}{R}\cos\theta\right)\cos\theta\,dA \\
&= \int_0^{2\pi}\int_0^{\pi} \left(-p_0 + \rho g R\cos\theta + \frac{3\mu v_\infty}{R}\cos\theta\right)\cos\theta\,(R^2\sin\theta\,d\theta\,d\phi) \\
&= \left(\int_0^{2\pi} R^2\,d\phi\right)\left(-p_0\underbrace{\int_0^{\pi}\cos\theta\sin\theta\,d\theta}_{=0} + \rho g R\underbrace{\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta}_{=2/3} + \frac{3\mu v_\infty}{R}\underbrace{\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta}_{=2/3}\right) \\
&= 2\pi R^2\left(\frac{2\rho g R}{3} + \frac{2\mu v_\infty}{R}\right) \\
&= \frac{4\pi R^3\rho g}{3} + 4\pi\mu R v_\infty
\end{aligned}$$

The force per unit area acting tangentially is due only to the viscous shear stress as shown below.




Integrate it over the surface area of the sphere to get the force. An extra factor of $\sin\theta$ is needed in the integrand to get the $z$-component of force specifically. As with $\tau_{rr}$, a minus sign has been placed in front of $\tau_{r\theta}$ because the fluid is in a region of greater $r$ acting on a surface of lesser $r$.

$$\begin{aligned}
(F_{\parallel})_z &= \int (-\tau_{r\theta})|_{r=R}\,\hat{\boldsymbol{\theta}}(\sin\theta)\cdot d\mathbf{A} \\
&= \int \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\bigg|_{r=R}\hat{\boldsymbol{\theta}}(\sin\theta)\cdot d\mathbf{A} \\
&= \int \mu\left[\frac{(r-R)v_\infty\sin\theta}{r^2}+\frac{(R-r)v_\infty\sin\theta}{r^2}\right]\bigg|_{r=R}\hat{\boldsymbol{\theta}}(\sin\theta)\cdot d\mathbf{A} \\
&= \int \mu(0)\hat{\boldsymbol{\theta}}(\sin\theta)\cdot d\mathbf{A} \\
&= 0
\end{aligned}$$

Therefore, the total force in the $z$-direction on the sphere is

$$\begin{aligned}
F_z &= (F_\perp)_z + (F_\parallel)_z \\
&= \frac{4}{3}\pi R^3\rho g + 4\pi\mu v_\infty R.
\end{aligned}$$

This first term is the buoyant force, and the second term is the kinetic force on the bubble due to the upward flow. By assuming the shear stress is zero, we obtain a kinetic force which is two-thirds that given by Stokes's law, which results from the no-slip assumption.