Problem 4B.5

Steady potential flow around a stationary sphere.[2] In Example 4.2-1 we worked through the creeping flow around a sphere. We now wish to consider the flow of an incompressible, inviscid fluid in irrotational flow around a sphere. For such a problem, we know that the velocity potential must satisfy Laplace's equation (see text after Eq. 4.3-11).

(a) State the boundary conditions for the problem.

(b) Give reasons why the velocity potential ϕ can be postulated to be of the form $\phi(r, \theta) = f(r)\cos\theta$.

(c) Substitute the trial expression for the velocity potential in (b) into Laplace's equation for the velocity potential.

(d) Integrate the equation obtained in (c) and obtain the function $f(r)$ containing two constants of integration; determine these constants from the boundary conditions and find

$$\phi = -v_\infty R\left[\left(\frac{r}{R}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^2\right]\cos\theta \tag{4B.5-1}$$

(e) Next show that

$$v_r = v_\infty\left[1 - \left(\frac{R}{r}\right)^3\right]\cos\theta \tag{4B.5-2}$$

$$v_\theta = -v_\infty\left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\sin\theta \tag{4B.5-3}$$

(f) Find the pressure distribution, and then show that at the sphere surface

$$\mathscr{P} - \mathscr{P}_\infty = \frac{1}{2}\rho v_\infty^2\left(1 - \frac{9}{4}\sin^2\theta\right) \tag{4B.5-4}$$

Solution

As in Problem 4B.3, the aim is to determine the velocity and pressure distributions around a stationary sphere immersed in a fluid that is flowing upward from the bottom. However, rather than assuming the flow is creeping, it is assumed here that the fluid is inviscid and the flow is irrotational.

[2] L. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, Boston, 2nd edition (1987), pp. 21–26, contains a good collection of potential-flow problems.

Use the same spherical coordinate system shown in Fig. 2.6-1.

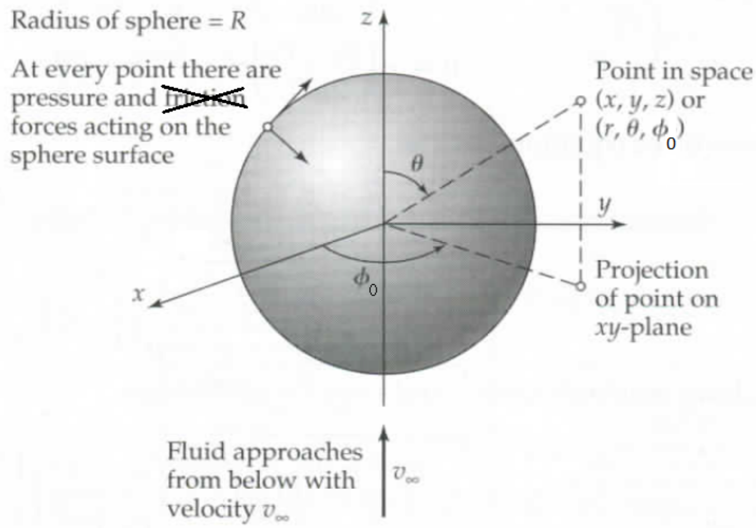

Fig. 2.6-1 Sphere of radius R around which a fluid is flowing. The coordinates r, θ, and ϕ_0 are shown. For more information on spherical coordinates, see Fig. A.8-2.

The fluid flow around the sphere is steady and symmetric with respect to the z-axis, so the velocity has no dependence on t or ϕ_0, the azimuthal angle.

$$\mathbf{v} = v_r(r,\theta)\hat{\mathbf{r}} + v_\theta(r,\theta)\hat{\boldsymbol{\theta}}$$

Because the flow is irrotational, the vorticity everywhere is zero.

$$\nabla \times \mathbf{v} = \mathbf{0}$$

Consequently, there exists a potential function $-\phi$ such that $\mathbf{v} = \nabla(-\phi) = -\nabla\phi$. The included minus sign is arbitrary mathematically, but physically it signifies that an increase in a fluid particle's potential is associated with a decrease in its velocity and vice-versa. The fluid is incompressible, so the continuity equation reduces to

$$\nabla \cdot \mathbf{v} = 0.$$

Substitute the formula for $\mathbf{v}$.

$$\nabla \cdot (-\nabla\phi) = 0$$

$$-\nabla^2\phi = 0$$

Multiplying both sides by -1, we see that the potential function satisfies the well-known Laplace equation.

$$\nabla^2\phi = 0$$

Associated with this PDE are three boundary conditions. The first comes from the fact that no mass passes through the sphere (that is, the sphere is impermeable). As a result, the radial velocity at $r = R$ is zero.

$$v_r(R,\theta) = 0$$

Since the fluid is inviscid, there is no frictional force on the sphere. In other words, the fluid slips on the sphere, resulting in a tangential velocity that is not zero at $r = R$. This is the second boundary condition.

$$v_\theta(R,\theta) \neq 0$$

The third boundary condition comes from the fact that far from the sphere's influence, the fluid velocity is $\mathbf{v} = v_\infty \hat{\mathbf{z}}$. Use formula A.6-33 on page 828 to write $\hat{\mathbf{z}}$ in terms of spherical unit vectors.

$$\lim_{r\to\infty} \mathbf{v} = v_\infty \hat{\mathbf{z}}$$

$$\lim_{r\to\infty} (-\nabla\phi) = v_\infty[(\cos\theta)\hat{\mathbf{r}} + (-\sin\theta)\hat{\boldsymbol{\theta}}]$$

$$-\lim_{r\to\infty} \nabla\phi = v_\infty \cos\theta\,\hat{\mathbf{r}} - v_\infty \sin\theta\,\hat{\boldsymbol{\theta}}$$

Use equations (D) and (E) on page 836 to expand the gradient operator in spherical coordinates.

$$-\lim_{r\to\infty}\left(\frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}}\right) = v_\infty \cos\theta\,\hat{\mathbf{r}} - v_\infty \sin\theta\,\hat{\boldsymbol{\theta}}$$

Multiply both sides by -1 and distribute the limit.

$$\lim_{r\to\infty}\frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \lim_{r\to\infty}\frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}} = -v_\infty \cos\theta\,\hat{\mathbf{r}} + v_\infty \sin\theta\,\hat{\boldsymbol{\theta}}$$

As a result,

$$\left.\begin{array}{lcl} \displaystyle\lim_{r\to\infty}\frac{\partial\phi}{\partial r} = -v_\infty\cos\theta & \to & \displaystyle\lim_{r\to\infty}\frac{\partial\phi}{\partial r} = -v_\infty\cos\theta \\ \displaystyle\lim_{r\to\infty}\frac{1}{r}\frac{\partial\phi}{\partial\theta} = v_\infty\sin\theta & \to & \displaystyle\lim_{r\to\infty}\frac{\partial\phi}{\partial\theta} = v_\infty r\sin\theta \end{array}\right\} \quad \Rightarrow \quad \lim_{r\to\infty}\phi(r,\theta) = -v_\infty r\cos\theta.$$

Because the Laplace equation is linear and homogeneous, there is a product solution of the form $\phi(r,\theta) = \xi(r)\Theta(\theta)$ by the method of separation of variables. In particular, based on the third boundary condition, it can be hypothesized that $\phi(r,\theta) = \xi(r)\cos\theta$. Substitute this formula into the Laplace equation to get an ODE for $\xi = \xi(r)$. Use formula (B) on page 836 to expand the Laplacian operator in spherical coordinates.

$$\nabla^2\phi = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}[\xi(r)\cos\theta]\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}[\xi(r)\cos\theta]\right) = 0$$

$$\frac{\cos\theta}{r^2}\frac{d}{dr}\left(r^2\frac{d\xi}{dr}\right) + \frac{\xi(r)}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta[-\sin\theta]\right) = 0$$

$$\frac{\cos\theta}{r^2}\frac{d}{dr}\left(r^2\frac{d\xi}{dr}\right) - \frac{\xi(r)}{r^2\sin\theta}\frac{d}{d\theta}(\sin^2\theta) = 0$$

$$\frac{\cos\theta}{r^2}\frac{d}{dr}\left(r^2\frac{d\xi}{dr}\right) - \frac{\xi(r)}{r^2\sin\theta}(2\sin\theta\cos\theta) = 0$$

$$\frac{\cos\theta}{r^2}\left[\frac{d}{dr}\left(r^2\frac{d\xi}{dr}\right) - 2\xi(r)\right] = 0$$

$$\frac{d}{dr}\left(r^2\frac{d\xi}{dr}\right) - 2\xi = 0$$

The ODE that ξ satisfies is then

$$r^2\frac{d^2\xi}{dr^2} + 2r\frac{d\xi}{dr} - 2\xi = 0.$$

It's equidimensional, so the solution is of the form $\xi = r^n$.

$$\xi = r^n \quad \rightarrow \quad \frac{d\xi}{dr} = nr^{n-1} \quad \rightarrow \quad \frac{d^2\xi}{dr^2} = n(n-1)r^{n-2}$$

Substitute these formulas into the ODE.

$$r^2 n(n-1)r^{n-2} + 2rnr^{n-1} - 2r^n = 0$$

$$n(n-1)r^n + 2nr^n - 2r^n = 0$$

Divide both sides by r^n.

$$n(n-1) + 2n - 2 = 0$$

Solve for n.

$$n^2 + n - 2 = 0$$

$$(n+2)(n-1) = 0$$

$$n = \{-2, 1\}$$

Two solutions to the ODE are $\xi = r^{-2}$ and $\xi = r^1 = r$. By the principle of superposition, the general solution is a linear combination of these two.

$$\xi(r) = C_1 r^{-2} + C_2 r$$

Noting that $\mathbf{v} = -\nabla\phi$, write the first boundary condition in terms of ξ

$$v_r(R,\theta) = 0 \quad \rightarrow \quad -\frac{\partial\phi}{\partial r}\bigg|_{r=R} = 0 \quad \rightarrow \quad -\frac{\partial}{\partial r}[\xi(r)\cos\theta]\bigg|_{r=R} = 0 \quad \rightarrow \quad -\cos\theta\frac{d\xi}{dr}\bigg|_{r=R} = 0$$

and then apply it to determine one of the constants.

$$\frac{d\xi}{dr}\bigg|_{r=R} = 0$$

$$-2C_1R^{-3} + C_2 = 0$$

$$C_2 = \frac{2C_1}{R^3}$$

So then

$$\begin{aligned}\xi(r) &= C_1 r^{-2} + \frac{2C_1}{R^3}r \\ &= \frac{2C_1}{R^3}r + \frac{C_1}{r^2}\end{aligned}$$

and

$$\begin{aligned}\phi(r,\theta) &= \xi(r)\cos\theta \\ &= \left(\frac{2C_1}{R^3}r + \frac{C_1}{r^2}\right)\cos\theta.\end{aligned}$$

Apply the third boundary condition to determine C_1.

$$\lim_{r\to\infty}\phi(r,\theta) = \frac{2C_1}{R^3} r\cos\theta = -v_\infty r\cos\theta$$

Match the coefficients and solve for C_1.

$$\frac{2C_1}{R^3} = -v_\infty$$

$$C_1 = -\frac{v_\infty R^3}{2}$$

Therefore, the velocity potential is

$$\begin{aligned}\phi(r,\theta) &= \left(\frac{2C_1}{R^3}r + \frac{C_1}{r^2}\right)\cos\theta \\ &= \left(-v_\infty r - \frac{v_\infty R^3}{2r^2}\right)\cos\theta \\ &= -v_\infty R\left(\frac{r}{R} + \frac{R^2}{2r^2}\right)\cos\theta \\ &= -v_\infty R\left[\left(\frac{r}{R}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^2\right]\cos\theta.\end{aligned}$$

Now that ϕ is known, the velocity is as well.

$$\begin{aligned}\mathbf{v} &= -\nabla\phi \\ &= -\left[\frac{\partial}{\partial r}\left(-v_\infty r - \frac{v_\infty R^3}{2r^2}\right)\cos\theta\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(-v_\infty r - \frac{v_\infty R^3}{2r^2}\right)\cos\theta\,\hat{\boldsymbol{\theta}}\right] \\ &= -\left[\left(-v_\infty + \frac{v_\infty R^3}{r^3}\right)\cos\theta\,\hat{\mathbf{r}} + \frac{1}{r}\left(-v_\infty r - \frac{v_\infty R^3}{2r^2}\right)(-\sin\theta)\,\hat{\boldsymbol{\theta}}\right] \\ &= -\left[\left(-v_\infty + \frac{v_\infty R^3}{r^3}\right)\cos\theta\,\hat{\mathbf{r}} + \left(v_\infty + \frac{v_\infty R^3}{2r^3}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right] \\ &= \left(v_\infty - \frac{v_\infty R^3}{r^3}\right)\cos\theta\,\hat{\mathbf{r}} - \left(v_\infty + \frac{v_\infty R^3}{2r^3}\right)\sin\theta\,\hat{\boldsymbol{\theta}} \\ &= v_\infty\left(1 - \frac{R^3}{r^3}\right)\cos\theta\,\hat{\mathbf{r}} - v_\infty\left(1 + \frac{R^3}{2r^3}\right)\sin\theta\,\hat{\boldsymbol{\theta}} \\ &= v_\infty\left[1 - \left(\frac{R}{r}\right)^3\right]\cos\theta\,\hat{\mathbf{r}} - v_\infty\left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\sin\theta\,\hat{\boldsymbol{\theta}}\end{aligned}$$

Therefore,

$$\begin{aligned}v_r(r,\theta) &= v_\infty\left[1 - \left(\frac{R}{r}\right)^3\right]\cos\theta \\ v_\theta(r,\theta) &= -v_\infty\left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\sin\theta\end{aligned}.$$

To obtain the pressure distribution, use the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

Since the fluid is inviscid, $\mu = 0$, and because the flow is steady, the time derivative vanishes.

$$\nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \rho\mathbf{g}$$

From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in spherical coordinates.

$$\rho\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} + \underbrace{\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi}}_{=0} - \frac{v_\theta^2 + \overbrace{v_\phi^2}^{=0}}{r}\right) = -\frac{\partial p}{\partial r} + \rho g_r$$

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \underbrace{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}}_{=0} + \frac{v_r v_\theta - \overbrace{v_\phi^2\cot\theta}^{=0}}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \rho g_\theta$$

$$\rho\left(\underbrace{v_r\frac{\partial v_\phi}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial\theta}}_{=0} + \underbrace{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}}_{=0} + \underbrace{\frac{v_\phi v_r + v_\theta v_\phi\cot\theta}{r}}_{=0}\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} + \rho g_\phi$$

Gravity points down in the negative z-direction, so $\mathbf{g} = -g\hat{\mathbf{z}} = -g(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$. This means that $g_r = -g\cos\theta$, $g_\theta = g\sin\theta$, and $g_\phi = 0$.

$$\rho\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} - \rho g\cos\theta$$

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \rho g\sin\theta$$

$$0 = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}$$

Multiply both sides of the third equation by $-r\sin\theta$ and factor each right side of the first two.

$$\rho\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta^2}{r}\right) = -\left(\frac{\partial p}{\partial r} + \rho g\cos\theta\right)$$

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\left(\frac{\partial p}{\partial\theta} - \rho g r\sin\theta\right)$$

$$0 = \frac{\partial p}{\partial\phi}$$

$$\rho\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta^2}{r}\right) = -\frac{\partial}{\partial r}(p + \rho g r\cos\theta)$$

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial}{\partial\theta}(p + \rho g r\cos\theta)$$

$$0 = \frac{\partial}{\partial\phi}(p + \rho g r\cos\theta)$$

Introduce the modified pressure function $\mathscr{P} = \mathscr{P}(r,\theta) = p(r,\theta) + \rho g r\cos\theta$.

$$\rho\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta^2}{r}\right) = -\frac{\partial\mathscr{P}}{\partial r}$$
$$\rho\left(v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial\mathscr{P}}{\partial\theta}$$
$$0 = \frac{\partial\mathscr{P}}{\partial\phi}$$

Solve for each of the modified pressure derivatives.

$$\frac{\partial\mathscr{P}}{\partial r} = -\rho\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta^2}{r}\right)$$
$$\frac{\partial\mathscr{P}}{\partial\theta} = -\rho r\left(v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r}\right)$$
$$\frac{\partial\mathscr{P}}{\partial\phi} = 0$$

Substitute the functions found for v_r and v_θ, evaluate the derivatives, and simplify each right side.

$$\frac{\partial\mathscr{P}}{\partial r} = \frac{3\rho v_\infty^2 R^3}{8}\left[\frac{5R^3}{r^7} - \frac{2}{r^4} + 3\left(\frac{R^3}{r^7} - \frac{2}{r^4}\right)\cos 2\theta\right] \tag{1}$$
$$\frac{\partial\mathscr{P}}{\partial\theta} = \frac{3\rho v_\infty^2 R^3}{8}\left(\frac{R^3}{r^6} - \frac{4}{r^3}\right)\sin 2\theta \tag{2}$$

Integrate both sides of equation (2) partially with respect to θ to get $\mathscr{P}$.

$$\mathscr{P}(r,\theta) = \frac{3\rho v_\infty^2 R^3}{16}\left(\frac{4}{r^3} - \frac{R^3}{r^6}\right)\cos 2\theta + f(r)$$

Differentiate both sides with respect to r.

$$\frac{\partial\mathscr{P}}{\partial r} = \frac{3\rho v_\infty^2 R^3}{8}\left(\frac{3R^3}{r^7} - \frac{6}{r^4}\right)\cos 2\theta + f'(r)$$

Comparing this to equation (1), we see that

$$f'(r) = \frac{3\rho v_\infty^2 R^3}{8}\left(\frac{5R^3}{r^7} - \frac{2}{r^4}\right).$$

Integrate both sides with respect to r, using $\mathscr{P}_\infty$ for the integration constant. This is the pressure far from the sphere at $z = r\cos\theta = 0$.

$$f(r) = \frac{3\rho v_\infty^2 R^3}{8}\left(\frac{2}{3r^3} - \frac{5R^3}{6r^6}\right) + \mathscr{P}_\infty$$

The modified pressure is then

$$\mathscr{P}(r,\theta) = \frac{3\rho v_\infty^2 R^3}{16}\left(\frac{4}{r^3} - \frac{R^3}{r^6}\right)\cos 2\theta + \frac{3\rho v_\infty^2 R^3}{8}\left(\frac{2}{3r^3} - \frac{5R^3}{6r^6}\right) + \mathscr{P}_\infty.$$

Evaluate it at the sphere surface $r = R$.

$$\begin{aligned}
\mathscr{P}(R,\theta) &= \frac{3\rho v_\infty^2 R^3}{16}\left(\frac{4}{R^3} - \frac{1}{R^3}\right)\cos 2\theta + \frac{3\rho v_\infty^2 R^3}{8}\left(\frac{2}{3R^3} - \frac{5}{6R^3}\right) + \mathscr{P}_\infty \\
&= \frac{9\rho v_\infty^2}{16}\cos 2\theta - \frac{\rho v_\infty^2}{16} + \mathscr{P}_\infty \\
&= \frac{9\rho v_\infty^2}{16}(1 - 2\sin^2\theta) - \frac{\rho v_\infty^2}{16} + \mathscr{P}_\infty \\
&= \frac{\rho v_\infty^2}{2} - \frac{9\rho v_\infty^2}{8}\sin^2\theta + \mathscr{P}_\infty \\
&= \frac{\rho v_\infty^2}{2}\left(1 - \frac{9}{4}\sin^2\theta\right) + \mathscr{P}_\infty
\end{aligned}$$

Therefore, at the sphere surface,

$$\mathscr{P} - \mathscr{P}_\infty = \frac{1}{2}\rho v_\infty^2\left(1 - \frac{9}{4}\sin^2\theta\right).$$