Problem 4B.7

Vortex flow.

(a) Show that the complex potential $w = (i\Gamma/2\pi)\ln z$ describes the flow in a vortex. Verify that the tangential velocity is given by $v_\theta = \Gamma/2\pi r$ and that $v_r = 0$. This type of flow is sometimes called a *free vortex*. Is this flow irrotational?

(b) Compare the functional dependence of v_θ on r in (a) with that which arose in Example 3.6-4. The latter kind of flow is sometimes called a *forced vortex*. Actual vortices, such as those that occur in a stirred tank, have a behavior intermediate between these two idealizations.

Solution

Part (a)

The logarithm makes things difficult, so take the derivative of w and write it in rectangular form instead of w. Set $z = re^{i\theta}$ rather than $z = x + iy$ because the velocity components are desired in terms of r and θ.

$$\begin{aligned}
\frac{dw}{dz} &= \frac{i\Gamma}{2\pi z} \\
&= \frac{i\Gamma}{2\pi r e^{i\theta}} \\
&= \frac{i\Gamma}{2\pi r} e^{-i\theta} \\
&= \frac{i\Gamma}{2\pi r}(\cos\theta - i\sin\theta) \\
&= \frac{\Gamma}{2\pi r}\sin\theta + i\frac{\Gamma}{2\pi r}\cos\theta \\
&= -v_x + i v_y
\end{aligned}$$

The derivative of w is known as the complex velocity, and its real and imaginary parts are $-v_x$ and v_y, respectively.

$$\begin{aligned}
v_x &= -\frac{\Gamma}{2\pi r}\sin\theta \\
v_y &= \frac{\Gamma}{2\pi r}\cos\theta
\end{aligned}$$

Use formulas A.6-13 and A.6-14 on page 827 to write the Cartesian unit vectors in terms of polar unit vectors and get v_r and v_θ.

$$\begin{aligned}
\mathbf{v} &= v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} \\
&= -\frac{\Gamma}{2\pi r}\sin\theta\,\hat{\mathbf{x}} + \frac{\Gamma}{2\pi r}\cos\theta\,\hat{\mathbf{y}} \\
&= -\frac{\Gamma}{2\pi r}\sin\theta(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) + \frac{\Gamma}{2\pi r}\cos\theta(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}) \\
&= -\cancel{\frac{\Gamma}{2\pi r}\sin\theta\cos\theta\,\hat{\mathbf{r}}} + \frac{\Gamma}{2\pi r}\sin^2\theta\,\hat{\boldsymbol{\theta}} + \cancel{\frac{\Gamma}{2\pi r}\cos\theta\sin\theta\,\hat{\mathbf{r}}} + \frac{\Gamma}{2\pi r}\cos^2\theta\,\hat{\boldsymbol{\theta}} \\
&= \frac{\Gamma}{2\pi r}\hat{\boldsymbol{\theta}}
\end{aligned}$$

Therefore, the components of velocity in polar coordinates are

$$v_r = 0$$
$$v_\theta = \frac{\Gamma}{2\pi r}.$$

In order to see what the flow looks like, we want to find the stream function $\psi(r, \theta)$. According to Table 4.2-1 on page 123, the stream function in polar coordinates satisfies

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}$$
$$v_\theta = \frac{\partial \psi}{\partial r}.$$

Substitute the formulas found for v_r and v_θ and solve for the derivatives.

$$\frac{\partial \psi}{\partial \theta} = 0 \tag{1}$$
$$\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r} \tag{2}$$

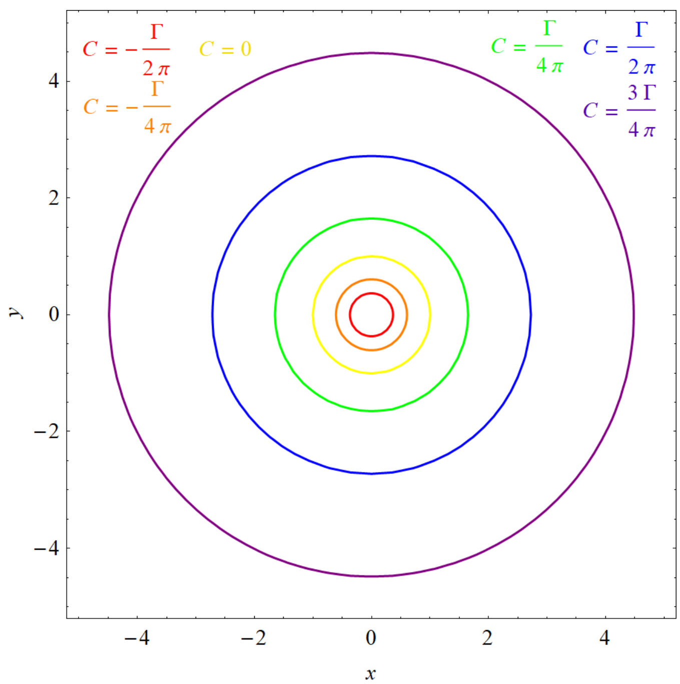
Integrate both sides of equation (2) partially with respect to r to get ψ.

$$\psi(r, \theta) = \frac{\Gamma}{2\pi} \ln r + f(\theta)$$

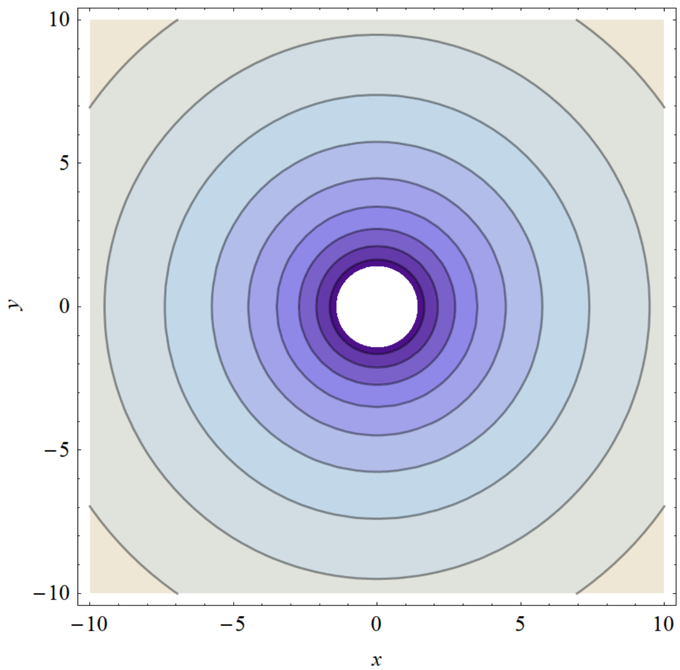
Set $f(\theta) = 0$ so that equation (1) is satisfied.

$$\psi(r, \theta) = \frac{\Gamma}{2\pi} \ln r$$

Now plot the streamlines $\psi(r, \theta) = C$ for various values of the constant C.



Instead of manually plotting the graphs for several values of C, it's more convenient to make a contour plot of the stream function for $\Gamma = 2\pi$.



$w = (i\Gamma/2\pi)\ln z$ does indeed describe the flow in a (free) vortex. To see if the flow is irrotational, calculate $\nabla \times \mathbf{v}$. Expand it in cylindrical coordinates by using formulas (G), (H), and (I) on page 834.

$$\begin{aligned}
\nabla \times \mathbf{v} &= \left(\underbrace{\frac{1}{r}\frac{\partial v_z}{\partial \theta}}_{=0} - \frac{\partial v_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\underbrace{\frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{\partial v_z}{\partial r}}_{=0}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \underbrace{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}_{=0}\right)\hat{\mathbf{z}} \\
&= -\frac{\partial}{\partial z}\left(\frac{\Gamma}{2\pi r}\right)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dr}\left[r\left(\frac{\Gamma}{2\pi r}\right)\right]\hat{\mathbf{z}} \\
&= \frac{1}{r}\frac{d}{dr}\left(\frac{\Gamma}{2\pi}\right)\hat{\mathbf{z}} \\
&= \mathbf{0}
\end{aligned}$$

Since the curl of $\mathbf{v}$ is zero, the flow is irrotational.

Part (b)

Example 3.6-4 dealt with a fluid in a cylindrical container rotating about its axis of symmetry with angular velocity Ω. There it was found that the fluid velocity was

$$\begin{aligned}
v_r &= 0 \\
v_\theta &= \Omega r \\
v_z &= 0.
\end{aligned}$$

To see what this flow looks like, calculate the stream function.

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}$$
$$v_\theta = \frac{\partial \psi}{\partial r}.$$

Substitute the formulas for v_r and v_θ and solve for the derivatives.

$$\frac{\partial \psi}{\partial \theta} = 0 \tag{3}$$
$$\frac{\partial \psi}{\partial r} = \Omega r \tag{4}$$

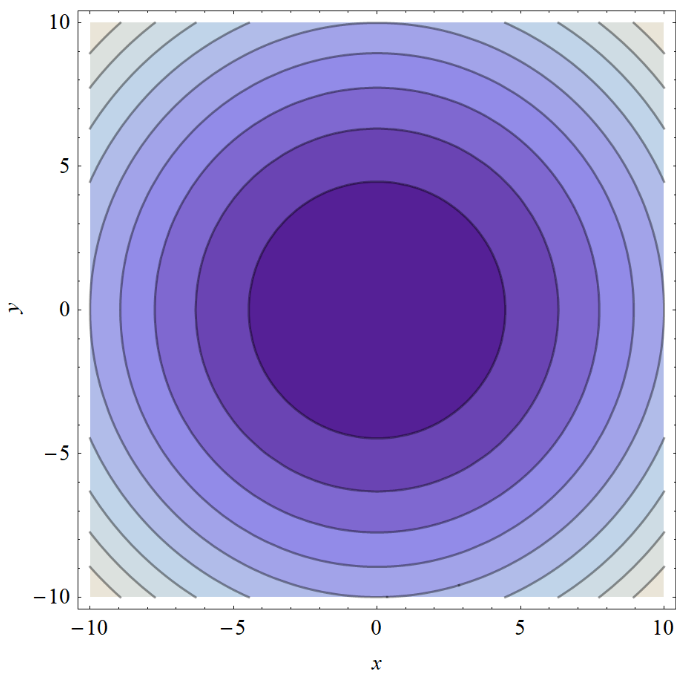
Integrate both sides of equation (4) partially with respect to r to get ψ.

$$\psi(r,\theta) = \frac{1}{2}\Omega r^2 + h(\theta)$$

Set $h(\theta) = 0$ to satisfy equation (3).

$$\psi(r,\theta) = \frac{1}{2}\Omega r^2$$

Make a contour plot of the stream function for $\Omega = 2$.



Comparing this plot with the previous one, we see that both have circular streamlines. The spacing between streamlines here decreases with increasing radius, whereas in the previous plot, the spacing between streamlines increases with increasing radius. This is due to the fact that v_θ varies linearly with r here and varies inversely with r in the previous one.

Below is a side-by-side comparison of the two stream functions for $\Gamma = 2\pi$ and $\Omega = 2$ in three-dimensional space.