Problem 4B.8

The flow field about a line source. Consider the symmetric radial flow of an incompressible, inviscid fluid outward from an infinitely long uniform source, coincident with the z-axis of a cylindrical coordinate system. Fluid is being generated at a volumetric rate Γ per unit length of source.

(a) Show that the Laplace equation for the velocity potential for this system is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0 \tag{4B.8-1}$$

(b) From this equation find the velocity potential, velocity, and pressure as functions of position:

$$\phi = -\frac{\Gamma}{2\pi}\ln r \qquad v_r = \frac{\Gamma}{2\pi r} \qquad \mathscr{P}_\infty - \mathscr{P} = \frac{\rho\Gamma^2}{8\pi^2 r^2} \tag{4B.8-2}$$

where $\mathscr{P}_\infty$ is the value of the modified pressure far away from the source.

(c) Discuss the applicability of the results in (b) to the flow field about a well drilled into a large body of porous rock.

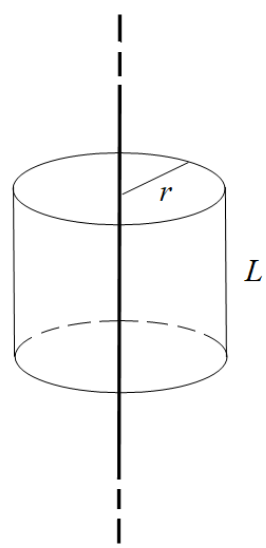
(d) Sketch the flow net of streamlines and equipotential lines.

Solution

Let the z-axis be coincident with the source and assume that the fluid flowing from it is steady and entirely radial.

$$\mathbf{v} = v_r(r)\hat{\mathbf{r}}$$

Consider a cylindrical shell of radius r and length L with an axis of symmetry along the source.



Calculate the volumetric flow through this shell.

$$\begin{aligned}\frac{dV}{dt} &= \mathbf{v}\cdot\mathbf{A} \\ &= v_r(r)\hat{\mathbf{r}}\cdot 2\pi r L\hat{\mathbf{r}} \\ &= 2\pi r L v_r(r)\end{aligned}$$

Divide both sides by L.

$$\frac{1}{L}\frac{dV}{dt} = 2\pi r v_r(r)$$

The volumetric flow rate per unit length is Γ.

$$\Gamma = 2\pi r v_r(r)$$

Therefore, the velocity is

$$\boxed{v_r(r) = \frac{\Gamma}{2\pi r}.}$$

Expand $\nabla \times \mathbf{v}$ in cylindrical coordinates by using formulas (G), (H), and (I) on page 834.

$$\begin{aligned}\nabla \times \mathbf{v} &= \bigg(\underbrace{\frac{1}{r}\frac{\partial v_z}{\partial \theta}}_{=0} - \underbrace{\frac{\partial v_\theta}{\partial z}}_{=0}\bigg)\hat{\mathbf{r}} + \bigg(\underbrace{\frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{\partial v_z}{\partial r}}_{=0}\bigg)\hat{\boldsymbol{\theta}} + \bigg(\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)}_{=0} - \underbrace{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}_{=0}\bigg)\hat{\mathbf{z}} \\ &= \mathbf{0}\end{aligned}$$

Since the vorticity is zero, there exists a potential function $-\phi$ such that $\mathbf{v} = \nabla(-\phi) = -\nabla\phi$. The included minus sign is arbitrary mathematically, but physically it signifies that an increase in a fluid particle's potential is associated with a decrease in its velocity and vice-versa. We could just solve for the potential by expanding the gradient operator using formulas (D), (E), and (F) on page 834.

$$\mathbf{v} = -\nabla\phi \quad \rightarrow \quad \frac{\Gamma}{2\pi r}\hat{\mathbf{r}} = -\frac{\partial \phi}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\boldsymbol{\theta}} - \frac{\partial \phi}{\partial z}\hat{\mathbf{z}} \quad \Rightarrow \quad \begin{cases}\dfrac{\partial \phi}{\partial r} = -\dfrac{\Gamma}{2\pi r} \\ \dfrac{\partial \phi}{\partial \theta} = 0 \\ \dfrac{\partial \phi}{\partial z} = 0\end{cases} \quad \Rightarrow \quad \phi(r) = -\frac{\Gamma}{2\pi}\ln r$$

But that would be too easy. To go along with the problem, just pretend we only know that $\mathbf{v} = v_r(r)\hat{\mathbf{r}}$. The fluid is incompressible, so the continuity equation reduces to

$$\nabla \cdot \mathbf{v} = 0.$$

Substitute the formula for $\mathbf{v}$.

$$\nabla \cdot (-\nabla\phi) = 0$$

$$-\nabla^2\phi = 0$$

Multiplying both sides by -1, we see that the potential function satisfies the well-known Laplace equation.

$$\nabla^2\phi = 0$$

Use formula (B) on page 834 to expand the Laplacian operator.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Because the velocity only depends on r, the velocity potential also only depends on r.

$$\mathbf{v} = -\nabla\phi \quad \rightarrow \quad v_r(r)\hat{\mathbf{r}} = -\frac{\partial \phi}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\boldsymbol{\theta}} - \frac{\partial \phi}{\partial z}\hat{\mathbf{z}} \quad \Rightarrow \quad \begin{cases}\dfrac{\partial \phi}{\partial \theta} = 0 \\ \dfrac{\partial \phi}{\partial z} = 0\end{cases} \quad \Rightarrow \quad \phi = \phi(r)$$

As a result, Laplace's equation reduces to an ODE.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0$$

Multiply both sides by r.

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{d\phi}{dr} = C_1$$

Divide both sides by r.

$$\frac{d\phi}{dr} = \frac{C_1}{r}$$

Integrate both sides with respect to r once more.

$$\phi(r) = C_1 \ln r + C_2$$

Unfortunately, without boundary conditions, C_1 and C_2 can't be found. Use the velocity found earlier and the fact that $\mathbf{v} = -\nabla\phi$ to determine C_1.

$$v_r(r)\hat{\mathbf{r}} = -\frac{d\phi}{dr}\hat{\mathbf{r}} \quad \rightarrow \quad \frac{\Gamma}{2\pi r}\hat{\mathbf{r}} = -\frac{C_1}{r}\hat{\mathbf{r}} \quad \Rightarrow \quad C_1 = -\frac{\Gamma}{2\pi}$$

C_2 remains arbitrary. Therefore, setting $C_2 = 0$,

$$\boxed{\phi(r) = -\frac{\Gamma}{2\pi}\ln r.}$$

To obtain the pressure distribution, use the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

Since the fluid is inviscid, $\mu = 0$, and because the flow is steady, the time derivative vanishes.

$$\nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \rho\mathbf{g}$$

From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(v_r\frac{\partial v_r}{\partial r} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{v_\theta^2}{r}}_{=0}\right) = -\frac{\partial p}{\partial r} + \rho g_r$$

$$\rho\left(\underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta$$

$$\rho\left(\underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial z} + \rho g_z$$

Gravity points down in the negative z-direction, so $\mathbf{g} = -g\hat{\mathbf{z}}$. That means $g_r = 0$, $g_\theta = 0$, and $g_z = -g$.

$$\begin{aligned} \rho v_r \frac{dv_r}{dr} &= -\frac{\partial p}{\partial r} \\ 0 &= -\frac{1}{r}\frac{\partial p}{\partial \theta} \\ 0 &= -\frac{\partial p}{\partial z} - \rho g \end{aligned}$$

Write each right side as a derivative.

$$\begin{aligned} \rho v_r \frac{dv_r}{dr} &= -\frac{\partial}{\partial r}(p + \rho g z) \\ 0 &= -\frac{1}{r}\frac{\partial}{\partial \theta}(p + \rho g z) \\ 0 &= -\frac{\partial}{\partial z}(p + \rho g z) \end{aligned}$$

Introduce the modified pressure function $\mathscr{P} = \mathscr{P}(r, z) = p(r, z) + \rho g z$.

$$\begin{aligned} \rho v_r \frac{dv_r}{dr} &= -\frac{\partial \mathscr{P}}{\partial r} \\ 0 &= -\frac{1}{r}\frac{\partial \mathscr{P}}{\partial \theta} \\ 0 &= -\frac{\partial \mathscr{P}}{\partial z} \end{aligned}$$

Solve for the derivatives.

$$\begin{aligned} \frac{\partial \mathscr{P}}{\partial r} &= -\rho v_r \frac{dv_r}{dr} \\ \frac{\partial \mathscr{P}}{\partial \theta} &= 0 \\ \frac{\partial \mathscr{P}}{\partial z} &= 0 \end{aligned}$$

These last two equations imply that the modified pressure function is only a function of r: $\mathscr{P} = \mathscr{P}(r)$.

$$\begin{aligned} \frac{d\mathscr{P}}{dr} &= -\rho v_r \frac{dv_r}{dr} \\ &= \frac{d}{dr}\left(-\frac{\rho}{2} v_r^2\right) \end{aligned}$$

Integrate both sides with respect to r, using $\mathscr{P}_\infty$ for the integration constant. This is the pressure far from the source at $z = 0$.

$$\begin{aligned} \mathscr{P}(r) &= -\frac{\rho}{2} v_r^2 + \mathscr{P}_\infty \\ &= -\frac{\rho}{2}\left(\frac{\Gamma}{2\pi r}\right)^2 + \mathscr{P}_\infty \\ &= -\frac{\rho \Gamma^2}{8\pi^2 r^2} + \mathscr{P}_\infty \end{aligned}$$

Therefore,

$$\boxed{\mathscr{P}_\infty - \mathscr{P} = \frac{\rho\Gamma^2}{8\pi^2 r^2}.}$$

According to Table 4.2-1 on page 123, the stream function in a cylindrical coordinate system with axial symmetry satisfies

$$\begin{aligned} v_z &= -\frac{1}{r}\frac{\partial\psi}{\partial r} \\ v_r &= \frac{1}{r}\frac{\partial\psi}{\partial z}. \end{aligned}$$

Substitute the formulas for v_r and v_z.

$$\begin{aligned} 0 &= -\frac{1}{r}\frac{\partial\psi}{\partial r} \\ \frac{\Gamma}{2\pi r} &= \frac{1}{r}\frac{\partial\psi}{\partial z}. \end{aligned}$$

Solve for the derivatives.

$$\begin{aligned} \frac{\partial\psi}{\partial r} &= 0 \\ \frac{\partial\psi}{\partial z} &= \frac{\Gamma}{2\pi} \end{aligned}$$

This first equation implies that ψ is only a function of z: $\psi = \psi(z)$.

$$\frac{d\psi}{dz} = \frac{\Gamma}{2\pi}$$

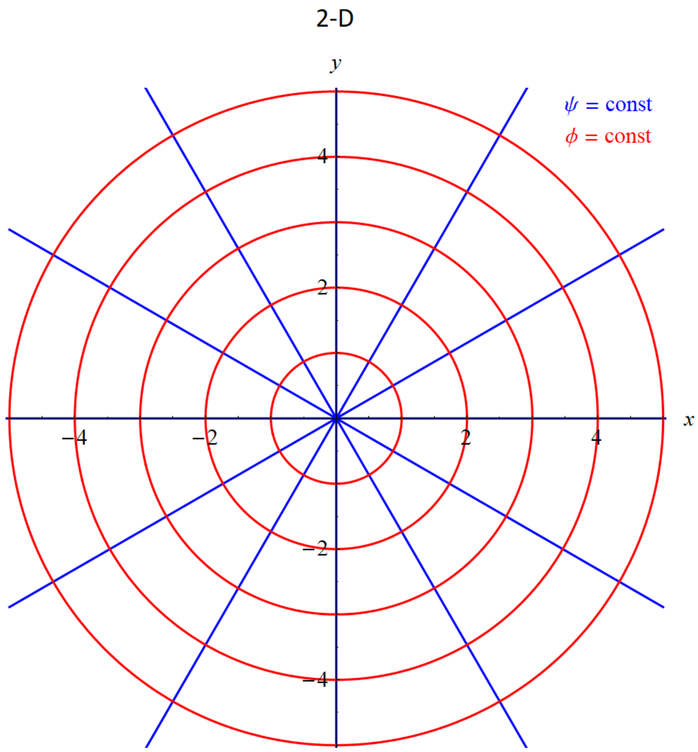
Integrate both sides with respect to z, setting the integration constant to zero.

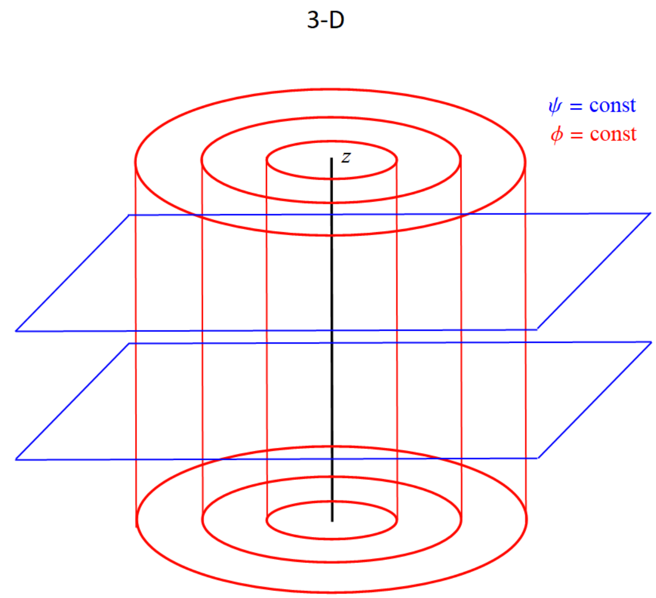
$$\psi(z) = \frac{\Gamma}{2\pi}z$$

To obtain the streamlines and equipotential lines, plot $\psi = C_3$ and $\phi = C_4$.

$$\begin{aligned} \psi(z) = C_3 \quad &\rightarrow \quad \frac{\Gamma}{2\pi}z = C_3 \quad &\rightarrow \quad z = \frac{2\pi C_3}{\Gamma} = \text{constant} \\ \phi(r) = C_4 \quad &\rightarrow \quad -\frac{\Gamma}{2\pi}\ln r = C_4 \quad &\rightarrow \quad r = \exp\left(-\frac{2\pi C_4}{\Gamma}\right) = \text{constant} \end{aligned}$$

Below are visuals in two dimensions and three dimensions.





The blue streamlines in 3-D indicate that the fluid only flows horizontally in planes. Only six blue streamlines going through the origin are drawn in 2-D; there should be infinitely many.

Whether or not the results in part (b) can be applied to a well depends on what kind of fluid we have and how deep the well extends. If we're talking about water or oil, then the assumption that the fluid is inviscid is not true, and the formula for pressure is expected to be inaccurate. Also, gravity would affect the motion of these liquids once the distance from the source gets high enough, so the assumption that $v_z = 0$ probably wouldn't hold either. For small r, the formula for velocity should be fine. If the fluid is a gas, then the assumption that it's incompressible won't hold, meaning that the formulas for the potential and velocity are expected to be inaccurate. Gases have low viscosity, though, so the formula involving pressure should be fine.