Problem 4C.1

Laminar entrance flow in a slit[3] (Fig. 4C.1). Estimate the velocity distribution in the entrance region of the slit shown in the figure. The fluid enters at $x = 0$ with $v_y = 0$ and $v_x = \langle v_x \rangle$, where $\langle v_x \rangle$ is the average velocity inside the slit. Assume that the velocity distribution in the entrance region $0 \le x \le L_e$ is

$$\frac{v_x}{v_e} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \text{(boundary layer region, } 0 \le y \le \delta\text{)} \qquad (4C.1\text{-}1)$$

$$\frac{v_x}{v_e} = 1 \qquad \text{(potential flow region, } \delta \le y \le B\text{)} \qquad (4C.1\text{-}2)$$

in which δ and v_e are functions of x, yet to be determined.

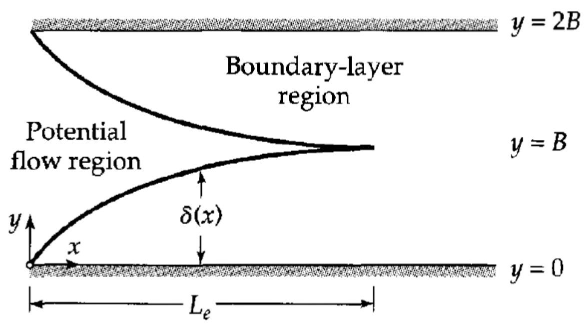


Fig. 4C.1. Entrance flow into a slit.

(a) Use the above two equations to get the mass flow rate w through an arbitrary cross section in the region $0 \le x \le L_e$. Then evaluate w from the inlet conditions and obtain

$$\frac{v_e(x)}{\langle v_x \rangle} = \frac{B}{B - \frac{1}{3}\delta(x)} \qquad (4C.1\text{-}3)$$

(b) Next use Eqs. 4.4-13, 4C.1-1, and 4C.1-2 with ∞ replaced by B (why?) to obtain a differential equation for the quantity $\Delta = \delta/B$:

$$\frac{6\Delta + 7\Delta^2}{(3-\Delta)^2}\frac{d\Delta}{dx} = 10\left(\frac{\nu}{\langle v_x \rangle B^2}\right) \qquad (4C.1\text{-}4)$$

(c) Integrate this equation with a suitable initial condition to obtain the following relation between the boundary-layer thickness and the distance down the duct:

$$\frac{\nu x}{\langle v_x \rangle B^2} = \frac{1}{10}\left[7\Delta + 48\ln\left(1 - \frac{1}{3}\Delta\right) + \frac{27\Delta}{3-\Delta}\right] \qquad (4C.1\text{-}5)$$

(d) Compute the entrance length L_e from Eq. 4C.1-5, where L_e is that value of x for which $\delta(x) = B$.

(e) Using potential flow theory, evaluate $\mathscr{P} - \mathscr{P}_0$ in the entrance region, where $\mathscr{P}_0$ is the value of the modified pressure at $x = 0$.

[3] A numerical solution to this problem using the Navier-Stokes equation has been given by Y. L. Wang and P. A. Longwell, *AIChE Journal*, **10**, 323–329 (1964).

Answers: **(d)** $L_e = 0.104\langle v_x\rangle B^2/\nu$; **(e)** $\mathscr{P} - \mathscr{P}_0 = \frac{1}{2}\rho\langle v_x\rangle^2\left[1 - \left(\frac{3}{3-\Delta}\right)^2\right]$

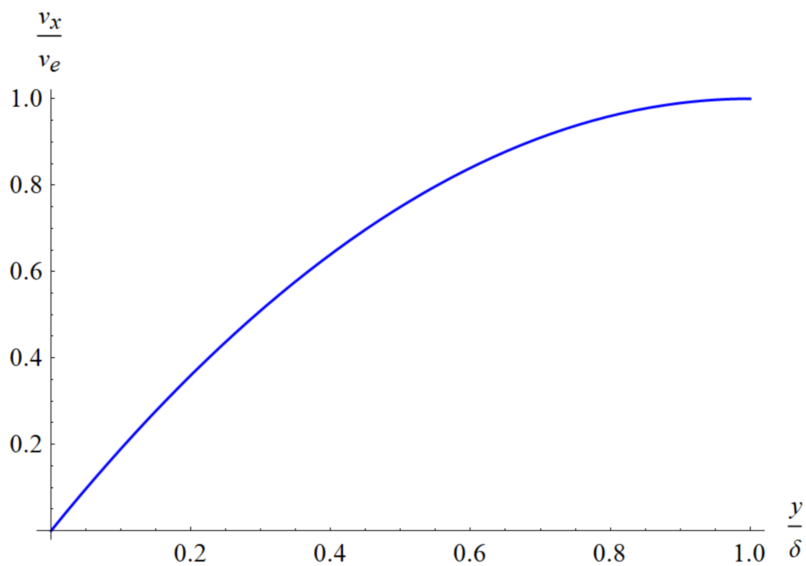
Solution

Problem 2B.3 dealt with the flow in a narrow slit between two parallel plates. Here in Problem 4C.1 the flow at the entrance to the slit will be analyzed. Fig. 4C.1 illustrates the situation for tangential laminar flow coming from the left. Two boundary layers form as a result, one due to the plate on the bottom and one due to the plate on top. By symmetry, the boundary layers end at $y = B$, half the length of the slit, and at $x = L_e$, an unknown for now. Past $x = L_e$, the familiar results from Problem 2B.3 apply. The velocity in the boundary layers is assumed to grow parabolically. At $y = \delta(x)$, the edge of the boundary layer, the fluid velocity is $v_x = v_e(x)$, the entrance velocity, and at $y = 0$ the fluid velocity is zero because the plate is stationary and the fluid is assumed not to slip on it. The postulated fluid velocity that satisfies these conditions is

$$\frac{v_x}{v_e} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \text{(boundary layer region, } 0 \le y \le \delta\text{)}$$

$$\frac{v_x}{v_e} = 1 \qquad \text{(potential flow region, } \delta \le y \le B\text{).}$$

Below is an illustration of the assumed velocity profile in the boundary layer region.



Part (a)

The volumetric flow rate through a cross-section in the slit is obtained by multiplying the x-component of velocity v_x by the cross-sectional area A that the fluid is flowing through.

$$\frac{dV}{dt} = v_x A$$

Since the velocity varies in the y-direction, integration, not multiplication, is required.

$$\frac{dV}{dt} = \int v_x \, dA$$

Suppose that W is the width of the slit in the z-direction. Then $dA = W\,dy$.

$$\frac{dV}{dt} = \int_0^{2B} v_x(x,y)(W\,dy)$$

Because of the slit's symmetry about the $y = B$ plane, the integral can be taken from 0 to B if a factor of 2 is placed in front.

$$\frac{dV}{dt} = 2\int_0^{B} v_x(x,y)(W\,dy)$$

To get the mass flow rate, multiply both sides by the fluid density ρ.

$$\rho\frac{dV}{dt} = 2\rho\int_0^{B} v_x(x,y)(W\,dy)$$

ρ is assumed to be constant, so it can be brought inside the derivative.

$$\frac{d(\rho V)}{dt} = 2\rho W\int_0^{B} v_x(x,y)\,dy$$

Density times volume is mass.

$$\begin{aligned}
\frac{dm}{dt} &= 2\rho W\int_0^{B} v_x(x,y)\,dy \\
&= 2\rho W\left[\int_0^{\delta} v_x(x,y)\,dy + \int_\delta^{B} v_x(x,y)\,dy\right] \\
&= 2\rho W\left\{\int_0^{\delta}\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] v_e(x)\,dy + \int_\delta^{B} v_e(x)\,dy\right\} \\
&= 2\rho W\left[v_e(x)\left(\frac{2}{\delta}\int_0^{\delta} y\,dy - \frac{1}{\delta^2}\int_0^{\delta} y^2\,dy\right) + v_e(x)\left(\int_\delta^{B} dy\right)\right] \\
&= 2\rho W v_e(x)\left[\left(\frac{2}{\delta}\frac{\delta^2}{2} - \frac{1}{\delta^2}\frac{\delta^3}{3}\right) + (B-\delta)\right] \\
&= 2\rho W v_e(x)\left[\left(\delta - \frac{\delta}{3}\right) + B - \delta\right] \\
&= 2\rho W v_e(x)\left(B - \frac{\delta}{3}\right)
\end{aligned}$$

This is the mass flow rate through some cross-section for $0 < x < L_e$. For the $x = 0$ cross-section in particular, $v_x = \langle v_x\rangle$, the (constant) average velocity in the slit downstream. The mass flow rate through the $x = 0$ cross-section is

$$\begin{aligned}
\frac{dV}{dt} &= v_x A \\
\frac{dV}{dt} &= \langle v_x\rangle A \\
\rho\frac{dV}{dt} &= \rho\langle v_x\rangle A \\
\frac{dm}{dt} &= \rho\langle v_x\rangle A \\
&= \rho\langle v_x\rangle[(2B)W] \\
&= 2\rho W B\langle v_x\rangle.
\end{aligned}$$

By the law of conservation of mass, matter is neither created nor destroyed. What goes through the cross-section at $x = 0$ must go through any cross-section in $0 < x < L_e$.

$$2\rho W B \langle v_x \rangle = 2\rho W v_e(x)\left(B - \frac{\delta}{3}\right)$$

Therefore,

$$\frac{v_e(x)}{\langle v_x \rangle} = \frac{B}{B - \frac{1}{3}\delta(x)}.$$

Part (b)

The von Kármán momentum balance is given by

$$\mu \frac{\partial v_x}{\partial y}\bigg|_{y=0} = \frac{d}{dx}\int_0^B \rho v_x (v_e - v_x)\, dy + \frac{dv_e}{dx}\int_0^B \rho (v_e - v_x)\, dy.$$

Here the upper limits of integration have been changed from ∞ to B because the boundary layer from the bottom plate only goes up that high. Substitute the velocity profile valid for $0 \le y \le \delta$ on the left side and simplify the right side.

$$\mu \frac{\partial}{\partial y}\left\{\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] v_e(x)\right\}\bigg|_{y=0} = \rho \frac{d}{dx}\int_0^B v_x(v_e - v_x)\, dy + \rho \frac{dv_e}{dx}\int_0^B (v_e - v_x)\, dy$$

$$\mu\left[\left(\frac{2}{\delta} - \frac{2y}{\delta^2}\right) v_e(x)\right]\bigg|_{y=0} = \rho \frac{d}{dx}\left[v_e(x)\int_0^B v_x(x,y)\, dy - \int_0^B [v_x(x,y)]^2\, dy\right] + \rho \frac{dv_e}{dx}\left[v_e(x)\int_0^B dy - \int_0^B v_x(x,y)\, dy\right]$$

$$\mu\left(\frac{2}{\delta}\right) v_e(x) = \rho \frac{d}{dx}\left\{v_e(x)\left[\int_0^\delta v_x(x,y)\, dy + \int_\delta^B v_x(x,y)\, dy\right] - \left[\int_0^\delta [v_x(x,y)]^2\, dy + \int_\delta^B [v_x(x,y)]^2\, dy\right]\right\} + \rho \frac{dv_e}{dx}\left\{v_e(x)\int_0^B dy - \left[\int_0^\delta v_x(x,y)\, dy + \int_\delta^B v_x(x,y)\, dy\right]\right\}$$

Now substitute the appropriate velocity profiles on the right side.

$$\begin{aligned}
\frac{2\mu}{\delta} v_e(x) &= \rho \frac{d}{dx}\left\{v_e(x)\left[\int_0^\delta \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] v_e(x)\, dy + \int_\delta^B v_e(x)\, dy\right]\right. \\
&\qquad \left. - \left[\int_0^\delta \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]^2 [v_e(x)]^2\, dy + \int_\delta^B [v_e(x)]^2\, dy\right]\right\} \\
&\quad + \rho \frac{dv_e}{dx}\left\{v_e(x)\int_0^B dy - \left\{\int_0^\delta \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] v_e(x)\, dy + \int_\delta^B v_e(x)\, dy\right\}\right\} \\
&= \rho \frac{d}{dx}\left\{v_e(x)\left[v_e(x)\int_0^\delta \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] dy + v_e(x)\int_\delta^B dy\right]\right. \\
&\qquad \left. - \left[[v_e(x)]^2\int_0^\delta \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]^2 dy + [v_e(x)]^2\int_\delta^B dy\right]\right\} \\
&\quad + \rho \frac{dv_e}{dx}\left\{v_e(x)\int_0^B dy - \left\{v_e(x)\int_0^\delta \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] dy + v_e(x)\int_\delta^B dy\right\}\right\}
\end{aligned}$$

Evaluate the integrals and simplify the right side.

$$\begin{aligned}\frac{2\mu}{\delta}v_e(x) &= \rho\frac{d}{dx}\left\{v_e(x)\left[v_e(x)\left(\frac{2\delta}{3}\right)+v_e(x)(B-\delta)\right]\right. \\ &\qquad\left.-\left[[v_e(x)]^2\left(\frac{8\delta}{15}\right)+[v_e(x)]^2(B-\delta)\right]\right\} \\ &\qquad+\rho\frac{dv_e}{dx}\left\{v_e(x)(B)-\left\{v_e(x)\left(\frac{2\delta}{3}\right)+v_e(x)(B-\delta)\right\}\right\} \\ &= \rho\frac{d}{dx}\left\{[v_e(x)]^2\frac{2\delta(x)}{15}\right\} \\ &\qquad+\rho\frac{dv_e}{dx}\left\{v_e(x)\frac{\delta(x)}{3}\right\} \\ &= \rho\left[2v_e(x)\frac{dv_e}{dx}\frac{2\delta(x)}{15}+[v_e(x)]^2\frac{2}{15}\frac{d\delta}{dx}\right]+\frac{\rho}{3}v_e(x)\frac{dv_e}{dx}\delta(x)\end{aligned}$$

Divide both sides by $\rho v_e(x)$ and use the kinematic viscosity ν for μ/ρ.

$$\frac{2\nu}{\delta(x)} = 2\frac{dv_e}{dx}\frac{2\delta(x)}{15}+v_e(x)\frac{2}{15}\frac{d\delta}{dx}+\frac{1}{3}\frac{dv_e}{dx}\delta(x)$$

$$\frac{2\nu}{\delta(x)} = \frac{3}{5}\frac{dv_e}{dx}\delta(x)+\frac{2}{15}v_e(x)\frac{d\delta}{dx}$$

Substitute the result from part (a) for $v_e(x)$.

$$\frac{2\nu}{\delta(x)} = \frac{3}{5}\frac{d}{dx}\left[\frac{B}{B-\frac{1}{3}\delta(x)}\langle v_x\rangle\right]\delta(x)+\frac{2}{15}\left[\frac{B}{B-\frac{1}{3}\delta(x)}\langle v_x\rangle\right]\frac{d\delta}{dx}$$

Multiply both sides by $5\delta(x)$.

$$10\nu = 3\frac{d}{dx}\left[\frac{B}{B-\frac{1}{3}\delta(x)}\langle v_x\rangle\right][\delta(x)]^2+\frac{2}{3}\left[\frac{B}{B-\frac{1}{3}\delta(x)}\langle v_x\rangle\right]\delta(x)\frac{d\delta}{dx}$$

Divide both sides by $\langle v_x\rangle B^2$.

$$\begin{aligned}10\left(\frac{\nu}{\langle v_x\rangle B^2}\right) &= 3\frac{d}{dx}\left[\frac{B}{B-\frac{1}{3}\delta(x)}\right]\frac{[\delta(x)]^2}{B^2}+\frac{2}{3}\left[\frac{B}{B-\frac{1}{3}\delta(x)}\right]\frac{\delta(x)}{B}\frac{1}{B}\frac{d\delta}{dx} \\ &= 3\frac{d}{dx}\left[\frac{3}{3-\frac{1}{B}\delta(x)}\right]\left[\frac{\delta(x)}{B}\right]^2+\frac{2}{3}\left[\frac{3}{3-\frac{1}{B}\delta(x)}\right]\frac{\delta(x)}{B}\frac{1}{B}\frac{d\delta}{dx}\end{aligned}$$

Make the substitution $\Delta = \delta/B$. Then $d\Delta/dx = (1/B)d\delta/dx$.

$$\begin{aligned}10\left(\frac{\nu}{\langle v_x\rangle B^2}\right) &= 3\frac{d}{dx}\left(\frac{3}{3-\Delta}\right)\Delta^2+\frac{2}{3}\left(\frac{3}{3-\Delta}\right)\Delta\frac{d\Delta}{dx} \\ &= 3\left[\frac{3}{(3-\Delta)^2}\frac{d\Delta}{dx}\right]\Delta^2+\frac{2}{3-\Delta}\Delta\frac{d\Delta}{dx} \\ &= \left[\frac{9\Delta^2}{(3-\Delta)^2}+\frac{2\Delta}{3-\Delta}\right]\frac{d\Delta}{dx}\end{aligned}$$

Combine the terms in square brackets.

$$10\left(\frac{\nu}{\langle v_x\rangle B^2}\right) = \left[\frac{9\Delta^2}{(3-\Delta)^2} + \frac{2\Delta(3-\Delta)}{(3-\Delta)^2}\right]\frac{d\Delta}{dx}$$
$$= \left[\frac{9\Delta^2 + 6\Delta - 2\Delta^2}{(3-\Delta)^2}\right]\frac{d\Delta}{dx}$$

Therefore, the ODE that Δ satisfies is

$$\frac{6\Delta + 7\Delta^2}{(3-\Delta)^2}\frac{d\Delta}{dx} = 10\left(\frac{\nu}{\langle v_x\rangle B^2}\right).$$

Part (c)

Solve this ODE by separating variables.

$$\frac{6\Delta + 7\Delta^2}{(3-\Delta)^2}d\Delta = 10\left(\frac{\nu}{\langle v_x\rangle B^2}\right)dx$$

Integrate both sides.

$$\int \frac{6\Delta + 7\Delta^2}{(\Delta-3)^2}\,d\Delta = \int 10\left(\frac{\nu}{\langle v_x\rangle B^2}\right)dx$$

Solve the integral on the left by using partial fraction decomposition.

$$\int\left[7 + \frac{48}{\Delta-3} + \frac{81}{(\Delta-3)^2}\right]d\Delta = 10\left(\frac{\nu}{\langle v_x\rangle B^2}\right)x + C_1$$

$$7\Delta + 48\ln|\Delta-3| - \frac{81}{\Delta-3} = 10\left(\frac{\nu}{\langle v_x\rangle B^2}\right)x + C_1$$

Notice from Fig. 4C.1 that the boundary layer thickness is zero at $x = 0$: $\delta(0) = 0$. That means $\Delta(0) = \delta(0)/B = 0$. Apply this boundary condition to determine C_1.

$$48\ln|-3| - \frac{81}{-3} = C_1$$

$$C_1 = \frac{81}{3} + 48\ln 3$$

$$7\Delta + 48\ln|\Delta-3| - \frac{81}{\Delta-3} = 10\left(\frac{\nu}{\langle v_x\rangle B^2}\right)x + \frac{81}{3} + 48\ln 3$$

Bring these last two terms on the right side to the left. Also, since $0 \le \Delta \le 1$, write $\Delta - 3$ as $3 - \Delta$.

$$7\Delta + 48\ln(3-\Delta) + \frac{81}{3-\Delta} - 48\ln 3 - \frac{81}{3} = 10\left(\frac{\nu}{\langle v_x\rangle B^2}\right)x$$

Combine the terms on the left side.

$$7\Delta + 48\ln\frac{3-\Delta}{3} + \frac{3(81) - 81(3-\Delta)}{3(3-\Delta)} = 10\left(\frac{\nu}{\langle v_x\rangle B^2}\right)x$$

$$7\Delta + 48\ln\left(1 - \frac{1}{3}\Delta\right) + \frac{81\Delta}{3(3-\Delta)} = 10\left(\frac{\nu}{\langle v_x\rangle B^2}\right)x$$

Therefore, dividing both sides by 10, Δ satisfies

$$\frac{\nu x}{\langle v_x\rangle B^2} = \frac{1}{10}\left[7\Delta + 48\ln\left(1 - \frac{1}{3}\Delta\right) + \frac{27\Delta}{3-\Delta}\right].$$

Part (d)

L_e is the value of x for which $\delta(x) = B$, or $\Delta(x) = 1$. Set $x = L_e$ and $\Delta = 1$ in this final result for part (c).

$$\frac{\nu L_e}{\langle v_x\rangle B^2} = \frac{1}{10}\left[7(1) + 48\ln\left(1 - \frac{1}{3}(1)\right) + \frac{27(1)}{3-1}\right]$$

Therefore, solving for L_e,

$$\begin{aligned} L_e &= \frac{1}{10}\left(\frac{41}{2} + 48\ln\frac{2}{3}\right)\frac{\langle v_x\rangle B^2}{\nu} \\ &\approx 0.104\frac{\langle v_x\rangle B^2}{\nu}. \end{aligned}$$

Part (e)

In order to get the pressure, use the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

Potential flow involves making the following critical assumptions: The fluid is inviscid ($\mu = 0$) and incompressible ($\nabla\cdot\mathbf{v} = 0$), and the flow is steady ($\partial\mathbf{v}/\partial t = \mathbf{0}$). The Navier-Stokes equation then reduces to

$$\nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \rho\mathbf{g}.$$

According to Appendix B.6 on page 848, it yields the following three scalar equations in Cartesian coordinates.

$$\begin{aligned} \rho\left(v_x\frac{\partial v_x}{\partial x} + \underbrace{v_y\frac{\partial v_x}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_x}{\partial z}}_{=0}\right) &= -\frac{\partial p}{\partial x} + \rho g_x \\ \rho\left(\underbrace{v_x\frac{\partial v_y}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_y}{\partial z}}_{=0}\right) &= -\frac{\partial p}{\partial y} + \rho g_y \\ \rho\left(\underbrace{v_x\frac{\partial v_z}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_z}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) &= -\frac{\partial p}{\partial z} + \rho g_z \end{aligned}$$

Gravity points to the right in the positive x-direction, so $\mathbf{g} = g\hat{\mathbf{x}}$. Compare Fig. 4C.1 with Fig. 2B.3 on page 63. That means $g_x = g$, $g_y = 0$, and $g_z = 0$.

$$\begin{aligned} \rho v_x\frac{\partial v_x}{\partial x} &= -\frac{\partial p}{\partial x} + \rho g \\ 0 &= -\frac{\partial p}{\partial y} \\ 0 &= -\frac{\partial p}{\partial z} \end{aligned}$$

Write each of the right sides as a derivatives.

$$\rho v_x \frac{\partial v_x}{\partial x} = -\frac{\partial}{\partial x}(p - \rho g x)$$
$$0 = -\frac{\partial}{\partial y}(p - \rho g x)$$
$$0 = -\frac{\partial}{\partial z}(p - \rho g x)$$

Introduce the modified pressure $\mathscr{P} = \mathscr{P}(x) = p(x) - \rho g x$.

$$\rho v_x \frac{\partial v_x}{\partial x} = -\frac{\partial \mathscr{P}}{\partial x}$$
$$0 = -\frac{\partial \mathscr{P}}{\partial y}$$
$$0 = -\frac{\partial \mathscr{P}}{\partial z}$$

Solve for the modified pressure derivatives.

$$\frac{\partial \mathscr{P}}{\partial x} = -\rho v_x \frac{\partial v_x}{\partial x}$$
$$\frac{\partial \mathscr{P}}{\partial y} = 0$$
$$\frac{\partial \mathscr{P}}{\partial z} = 0$$

These last two equations imply that the modified pressure is only a function of x: $\mathscr{P} = \mathscr{P}(x)$.

$$\frac{d\mathscr{P}}{dx} = -\rho v_x \frac{\partial v_x}{\partial x}$$

In the potential flow region ($\delta \le y \le B$), $v_x = v_e(x)$.

$$\frac{d\mathscr{P}}{dx} = -\rho v_e \frac{dv_e}{dx}$$
$$= \frac{d}{dx}\left\{-\frac{\rho}{2}[v_e(x)]^2\right\}$$

Integrate both sides with respect to x.

$$\int_0^x \frac{d\mathscr{P}}{dx}\,dx = \int_0^x \frac{d}{dx}\left\{-\frac{\rho}{2}[v_e(x)]^2\right\}dx$$
$$\mathscr{P}(x) - \mathscr{P}(0) = -\frac{\rho}{2}[v_e(x)]^2 + \frac{\rho}{2}[v_e(0)]^2$$

Note that $\langle v_x \rangle$ and $\mathscr{P}_0$ are the velocity and the pressure at $x = 0$, respectively.

$$\mathscr{P}(x) - \mathscr{P}_0 = -\frac{\rho}{2}[v_e(x)]^2 + \frac{\rho}{2}\langle v_x \rangle^2$$

Substitute the final result of part (a) for $v_e(x)$.

$$\mathscr{P}(x) - \mathscr{P}_0 = -\frac{\rho}{2}\left[\frac{B}{B - \frac{1}{3}\delta(x)}\langle v_x \rangle\right]^2 + \frac{\rho}{2}\langle v_x \rangle^2$$
$$= -\frac{1}{2}\rho\langle v_x \rangle^2\left[\frac{3}{3 - \frac{1}{B}\delta(x)}\right]^2 + \frac{1}{2}\rho\langle v_x \rangle^2$$

Replace $\delta(x)/B$ with Δ.

$$\mathscr{P}(x) - \mathscr{P}_0 = -\frac{1}{2}\rho\langle v_x\rangle^2\left(\frac{3}{3-\Delta}\right)^2 + \frac{1}{2}\rho\langle v_x\rangle^2$$

Therefore,

$$\mathscr{P} - \mathscr{P}_0 = \frac{1}{2}\rho\langle v_x\rangle^2\left[1 - \left(\frac{3}{3-\Delta}\right)^2\right].$$