Problem 4C.2

Torsional oscillatory viscometer (Fig. 4C.2). In the torsional oscillatory viscometer, the fluid is placed between a "cup" and "bob" as shown in the figure. The cup is made to undergo small sinusoidal oscillations in the tangential direction. This motion causes the bob, suspended by a torsion wire, to oscillate with the same frequency, but with a different amplitude and phase.

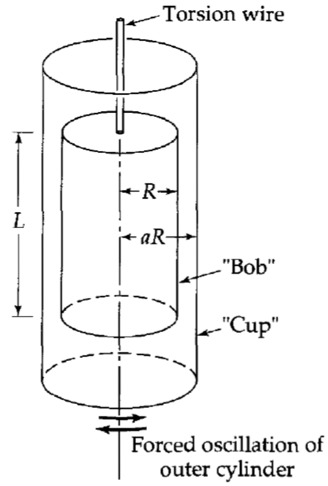


Fig. 4C.2. Sketch of a torsional oscillatory viscometer.

The amplitude ratio (ratio of amplitude of output function to input function) and phase shift both depend on the viscosity of the fluid and hence can be used for determining the viscosity. It is assumed throughout that the oscillations are of *small* amplitude. Then the problem is a linear one, and it can be solved either by Laplace transform or by the method outlined in this problem.

(a) First, apply Newton's second law of motion to the cylindrical bob for the special case that the annular space is completely evacuated. Show that the natural frequency of the system is $\omega_0 = \sqrt{k/I}$, in which I is the moment of inertia of the bob, and k is the spring constant for the torsion wire.

(b) Next, apply Newton's second law when there is a fluid of viscosity μ in the annular space. Let θ_R be the angular displacement of the bob at time t, and v_θ be the tangential velocity of the fluid as a function of r and t. Show that the equation of motion of the bob is

$$\text{(Bob)} \qquad I\frac{d^2\theta_R}{dt^2} = -k\theta_R + (2\pi RL)(R)\left(\mu r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right)\bigg|_{r=R} \qquad \text{(4C.2-1)}$$

If the system starts from rest, we have the initial conditions

$$\text{I.C.:} \qquad \text{at } t=0, \qquad \theta_R = 0 \quad \text{and} \quad \frac{d\theta_R}{dt} = 0 \qquad \text{(4C.2-2)}$$

(c) Next, write the equation of motion for the fluid along with the relevant initial and boundary conditions:

$$\text{(Fluid)} \qquad \rho\frac{\partial v_\theta}{\partial t} = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) \tag{4C.2-3}$$

$$\text{I.C.:} \qquad \text{at } t = 0, \qquad v_\theta = 0 \tag{4C.2-4}$$

$$\text{B.C. 1:} \qquad \text{at } r = R, \qquad v_\theta = R\frac{d\theta_R}{dt} \tag{4C.2-5}$$

$$\text{B.C. 2:} \qquad \text{at } r = aR, \qquad v_\theta = aR\frac{d\theta_{aR}}{dt} \tag{4C.2-6}$$

The function $\theta_{aR}(t)$ is a specified sinusoidal function (the "input"). Draw a sketch showing θ_{aR} and θ_R as functions of time, and defining the *amplitude ratio* and the *phase shift*.

(d) Simplify the starting equations, Eqs. 4C.2-1 to 6, by making the assumption that a is only slightly greater than unity, so that the curvature may be neglected (the problem can be solved without making this assumption[4]). This suggests that a suitable dimensionless distance variable is $x = (r - R)/[(a - 1)R]$. Recast the entire problem in dimensionless quantities in such a way that $1/\omega_0 = \sqrt{I/k}$ is used as a characteristic time, and so that the viscosity appears in just one dimensionless group. The only choice turns out to be:

$$\text{time:} \qquad \tau = \sqrt{\frac{k}{I}}t \tag{4C.2-7}$$

$$\text{velocity:} \qquad \phi = \frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}v_\theta \tag{4C.2-8}$$

$$\text{viscosity:} \qquad M = \frac{\mu/\rho}{(a-1)^2R^2}\sqrt{\frac{I}{k}} \tag{4C.2-9}$$

$$\text{reciprocal of moment of inertia:} \qquad A = \frac{2\pi R^4 L\rho(a-1)}{I} \tag{4C.2-10}$$

Show that the problem can now be restated as follows:

$$\text{(bob)} \qquad \frac{d^2\theta_R}{d\tau^2} = -\theta_R + M\left(\frac{\partial \phi}{\partial x}\right)\bigg|_{x=0} \qquad \text{at } \tau = 0,\ \theta_R = 0;\ d\theta_R/d\tau = 0 \tag{4C.2-11}$$

$$\text{(fluid)} \qquad \frac{\partial \phi}{\partial \tau} = M\frac{\partial^2 \phi}{\partial x^2} \qquad \begin{cases} \text{at } \tau = 0, & \phi = 0 \\ \text{at } x = 0, & \phi = A(d\theta_R/d\tau) \\ \text{at } x = 1, & \phi = A(d\theta_{aR}/d\tau) \end{cases} \tag{4C.2-12}$$

[4] H. Markovitz, *J. Appl. Phys.*, **23**, 1070–1077 (1952) has solved the problem without assuming a small spacing between the cup and bob. The cup-and-bob instrument has been used by L. J. Wittenberg, D. Ofte, and C. F. Curtiss, *J. Chem. Phys.*, **48**, 3253–3260 (1968), to measure the viscosity of liquid plutonium alloys.

(e) Obtain the "sinusoidal steady-state" solution by taking the input function θ_{aR} (the displacement of the cup) to be of the form

$$\theta_{aR}(\tau) = \theta^{\circ}_{aR}\Re\{e^{i\varpi\tau}\} \qquad (\theta^{\circ}_{aR} \text{ is real}) \tag{4C.2-13}$$

in which $\varpi = \omega/\omega_0 = \omega\sqrt{I/k}$ is a dimensionless frequency. Then postulate that the bob and fluid motions will also be sinusoidal, but with different amplitudes and phases:

$$\theta_R(\tau) = \Re\{\theta^{\circ}_R e^{i\varpi\tau}\} \qquad (\theta^{\circ}_R \text{ is complex}) \tag{4C.2-14}$$

$$\phi(x,\tau) = \Re\{\phi^{\circ}(x)e^{i\varpi\tau}\} \qquad (\phi^{\circ}(x) \text{ is complex}) \tag{4C.2-15}$$

Verify that the amplitude ratio is given by $|\theta^{\circ}_R|/\theta^{\circ}_{aR}$, where $|\cdots|$ indicates the absolute magnitude of a complex quantity. Further show that the phase angle α is given by $\tan\alpha = \Im\{\theta^{\circ}_R\}/\Re\{\theta^{\circ}_R\}$, where $\Re$ and $\Im$ stand for the real and imaginary parts, respectively.

(f) Substitute the postulated solutions of (e) into the equations in (d) to obtain equations for the complex amplitudes θ°_R and $\phi^{\circ}(x)$.

(g) Solve the equation for $\phi^{\circ}(x)$ and verify that

$$\left.\frac{d\phi^{\circ}}{dx}\right|_{x=0} = -\frac{A(i\varpi)^{3/2}}{\sqrt{M}}\left(\frac{\theta^{\circ}_R\cosh\sqrt{i\varpi/M} - \theta^{\circ}_{aR}}{\sinh\sqrt{i\varpi/M}}\right) \tag{4C.2-16}$$

(h) Next, solve the θ°_R equation to obtain

$$\frac{\theta^{\circ}_R}{\theta^{\circ}_{aR}} = \frac{AMi\varpi}{(1-\varpi^2)\dfrac{\sinh\sqrt{i\varpi/M}}{\sqrt{i\varpi/M}} + AMi\varpi\cosh\sqrt{i\varpi/M}} \tag{4C.2-17}$$

from which the amplitude ratio $|\theta^{\circ}_R|/\theta^{\circ}_{aR}$ and phase shift α can be found.

(i) For high-viscosity fluids, we can seek a power series by expanding the hyperbolic functions in Eq. 4C.2-17 to get a power series in $1/M$. Show that this leads to

$$\frac{\theta^{\circ}_{aR}}{\theta^{\circ}_R} = 1 + \frac{i}{M}\left(\frac{\varpi^2-1}{A\varpi} + \frac{\varpi}{2}\right) - \frac{1}{M^2}\left(\frac{\varpi^2-1}{6A} + \frac{\varpi^2}{24}\right) + O\left(\frac{1}{M^3}\right) \tag{4C.2-18}$$

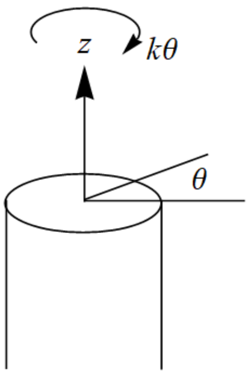
From this, find the amplitude ratio and the phase angle.

(j) Plot $|\theta^{\circ}_R|/\theta^{\circ}_{aR}$ versus ϖ for $\mu/\rho = 10^{-2}$ cm^2/s, $L = 25$ cm, $R = 5.5$ cm, $I = 2500$ g cm^2, $k = 4\times10^6$ dyn cm, $\rho = 1$ g/cm^3, $(a-1)R = 10^{-2}$ cm. Where is the maximum in the curve?

Solution

Part (a)

Suppose the bob is displaced an angle θ from equilibrium. Since there is no fluid in the annular space, the only torque acting on the bob is the one from the torsion wire. Setup a cylindrical coordinate system with the z-axis lying along the bob's axis of symmetry and draw a free-body diagram.



Apply the rotational analog of Newton's second law.

$$\sum \mathbf{T} = I\boldsymbol{\alpha}$$

Take the sum of the torques about the z-axis.

$$\sum T_z = I\alpha_z$$

The torsion wire applies a torque of magnitude $k\theta$ that opposes the motion, so a minus sign is needed.

$$-k\theta = I\alpha_z$$

Angular acceleration is the second derivative of angular position.

$$-k\theta = I\frac{d^2\theta}{dt^2}$$

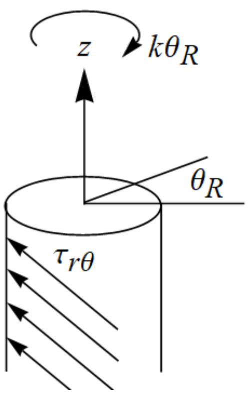
Therefore, the bob's equation of motion is

$$\frac{d^2\theta}{dt^2} = -\frac{k}{I}\theta$$

in the absence of fluid in the annular space. This is the ODE for simple harmonic motion, and by inspection the natural frequency of oscillation is $\omega_0 = \sqrt{k/I}$.

Part (b)

Now assume a fluid of constant mass density ρ and viscosity μ is present in the annular space. In response to an angular displacement θ_R, there is a torque from the torsion wire and a torque from the fluid's shearing stress on the bob's lateral surface. Use the same cylindrical coordinate system as before and draw a free-body diagram.



Apply the rotational analog of Newton's second law.

$$\sum \mathbf{T} = I\boldsymbol{\alpha}$$

Take the sum of the torques about the z-axis.

$$\sum T_z = I\alpha_z$$

From part (a) the torque from the torsion wire is $-k\theta_R$. To get the force due to the shearing stress, an integral needs to be taken of $-(-\tau_{r\theta})$ over the bob's lateral surface. Including a factor of R, the moment arm, in the integrand makes this integral a torque. Note that one minus sign is needed in front of $\tau_{r\theta}$ because the fluid is in a region of higher r acting on a surface of lower r, and a second minus sign is needed because the stress is acting in the negative θ-direction. A third minus sign is needed in front of the integral because, like the torsion wire, the stress due to viscosity resists the bob's motion.

$$-k\theta_R - \int R[-(-\tau_{r\theta})]\big|_{r=R}\, dA = I\alpha_z$$

$$-k\theta_R - \int_0^L R\tau_{r\theta}\big|_{r=R}\,(2\pi R\, dz) = I\alpha_z$$

$$-k\theta_R - 2\pi R^2 \int_0^L \tau_{r\theta}\big|_{r=R}\, dz = I\alpha_z$$

Use the formula for $\tau_{r\theta}$ on page 844.

$$-k\theta_R + 2\pi R^2 \int_0^L \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]\bigg|_{r=R} dz = I\alpha_z$$

The fluid flow is symmetric about the z-axis, so there's no variation in θ.

$$-k\theta_R + 2\pi R^2 \int_0^L \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]\bigg|_{r=R} dz = I\alpha_z$$

Evaluate the integral, assuming that v_θ does not depend on z.

$$-k\theta_R + 2\pi R^2 \int_0^L \mu R \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\bigg|_{r=R} dz = I\alpha_z$$

$$-k\theta_R + 2\pi R^3 \mu L \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\bigg|_{r=R} = I\alpha_z$$

Therefore, the bob's equation of motion is

$$-k\theta_R + 2\pi R^3 \mu L \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\bigg|_{r=R} = I\frac{d^2\theta_R}{dt^2}$$

in the presence of fluid in the annular space. If the bob is at rest initially, then there are two zero initial conditions associated with it.

$$\theta_R(0) = 0$$
$$\frac{d\theta_R}{dt}(0) = 0$$

Part (c)

In order to solve the bob's equation of motion, another equation that involves v_θ is needed. Assume that the fluid flows only in the θ-direction and that the velocity varies as a function of radius and time. Provided that the oscillations are small, this is reasonable.

$$\mathbf{v} = v_\theta(r,t)\hat{\boldsymbol{\theta}}$$

If we assume the fluid does not slip on the walls, then it has the bob's velocity at $r = R$ and the cup's velocity at $r = aR$. The tangential velocity is obtained by multiplying the angular velocity by the distance from the axis of rotation (the moment arm).

Boundary Condition 1: $\quad v_\theta(R,t) = R\frac{d\theta_R}{dt}$

Boundary Condition 2: $\quad v_\theta(aR,t) = aR\frac{d\theta_{aR}}{dt}$

The fluid starts at rest initially, so the initial condition is

Initial Condition: $\quad v_\theta(r,0) = 0.$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Because the fluid density is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

Expand the left side in cylindrical coordinates.

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0$$

This doesn't tell us anything. On the other hand, the equation of motion for the fluid results by considering a momentum balance over a volume element that the fluid is flowing through. Because the fluid viscosity is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_r}_{=0}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\right) = \underbrace{-\frac{1}{r}\frac{\partial p}{\partial \theta}}_{=0} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_\theta}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

The relevant equation for the velocity is the θ-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_\theta(r,t)\hat{\boldsymbol{\theta}}$.

$$\rho\frac{\partial v_\theta}{\partial t} = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)$$

Divide both sides by μ and expand the right side.

$$\frac{\rho}{\mu}\frac{\partial v_\theta}{\partial t} = \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} \tag{1}$$

Because this PDE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $f(r,t)$ is defined here as

$$F(r,s) = \mathcal{L}\{f(r,t)\} = \int_0^\infty e^{-st} f(r,t)\,dt,$$

and as a result, the derivatives transform as follows.

$$\mathcal{L}\left\{\frac{\partial f}{\partial t}(r,t)\right\} = sF(r,s) - f(r,0)$$
$$\mathcal{L}\left\{\frac{\partial^2 f}{\partial t^2}(r,t)\right\} = s^2F(r,s) - sf(r,0) - \frac{\partial f}{\partial t}(r,0)$$
$$\mathcal{L}\left\{\frac{\partial^n f}{\partial r^n}(r,t)\right\} = \frac{d^nF}{dr^n}(r,s)$$

Take the Laplace transform of both sides of equation (1).

$$\mathcal{L}\left\{\frac{\rho}{\mu}\frac{\partial v_\theta}{\partial t}\right\} = \mathcal{L}\left\{\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right\}$$

Use the fact that the transform is a linear operator.

$$\frac{\rho}{\mu}\mathcal{L}\left\{\frac{\partial v_\theta}{\partial t}\right\} = \mathcal{L}\left\{\frac{\partial^2 v_\theta}{\partial r^2}\right\} + \frac{1}{r}\mathcal{L}\left\{\frac{\partial v_\theta}{\partial r}\right\} - \frac{\mathcal{L}\{v_\theta\}}{r^2}$$

Transform the derivatives.

$$\frac{\rho}{\mu}[sV_\theta(r,s) - \underbrace{v_\theta(r,0)}_{=0}] = \frac{d^2V_\theta}{dr^2} + \frac{1}{r}\frac{dV_\theta}{dr} - \frac{V_\theta}{r^2}$$

By using the Laplace transform, the PDE in equation (1) has been reduced to an ODE. Bring all terms to one side.

$$\frac{d^2V_\theta}{dr^2} + \frac{1}{r}\frac{dV_\theta}{dr} - \frac{V_\theta}{r^2} - \frac{\rho s}{\mu}V_\theta(r,s) = 0$$

Multiply both sides by r^2 and factor V_θ.

$$r^2\frac{d^2V_\theta}{dr^2} + r\frac{dV_\theta}{dr} - \left(1 + \frac{\rho s}{\mu}r^2\right)V_\theta(r,s) = 0$$

Make the substitution,

$$w = \sqrt{\frac{\rho s}{\mu}}r.$$

The ODE then becomes

$$\frac{\mu}{\rho s}w^2\frac{d^2V_\theta}{dr^2} + \sqrt{\frac{\mu}{\rho s}}w\frac{dV_\theta}{dr} - (1+w^2)V_\theta(w,s) = 0.$$

Now use the chain rule to determine how the derivatives are written in terms of this new variable.

$$\frac{dV_\theta}{dr} = \frac{dV_\theta}{dw}\frac{dw}{dr} = \frac{dV_\theta}{dw}\sqrt{\frac{\rho s}{\mu}}$$

$$\frac{d^2V_\theta}{dr^2} = \frac{d}{dr}\left(\frac{dV_\theta}{dr}\right) = \frac{dw}{dr}\frac{d}{dw}\left(\frac{dV_\theta}{dw}\sqrt{\frac{\rho s}{\mu}}\right) = \sqrt{\frac{\rho s}{\mu}}\frac{d}{dw}\left(\frac{dV_\theta}{dw}\sqrt{\frac{\rho s}{\mu}}\right) = \frac{\rho s}{\mu}\frac{d^2V_\theta}{dw^2}$$

Substitute these formulas into the ODE.

$$\frac{\mu}{\rho s}w^2\left(\frac{\rho s}{\mu}\frac{d^2V_\theta}{dw^2}\right) + \sqrt{\frac{\mu}{\rho s}}w\left(\frac{dV_\theta}{dw}\sqrt{\frac{\rho s}{\mu}}\right) - (1+w^2)V_\theta(w,s) = 0$$

Therefore, the transformed fluid velocity satisfies the modified Bessel equation of first order.

$$w^2\frac{d^2V_\theta}{dw^2} + w\frac{dV_\theta}{dw} - (1+w^2)V_\theta(w,s) = 0$$

The general solution is a linear combination of I_1 and K_1, the first-order modified Bessel functions of the first and second kind, respectively.

$$V_\theta(w,s) = C_1(s)I_1(w) + C_2(s)K_1(w)$$

Change back to r now that the solution is known.

$$V_\theta(r,s) = C_1(s)I_1\left(\sqrt{\frac{\rho s}{\mu}}r\right) + C_2(s)K_1\left(\sqrt{\frac{\rho s}{\mu}}r\right)$$

In order to determine the arbitrary functions, $C_1(s)$ and $C_2(s)$, the boundary conditions for v_θ need to be transformed.

$$v_\theta(R,t) = R\frac{d\theta_R}{dt} \quad \rightarrow \quad \mathcal{L}\{v_\theta(R,t)\} = \mathcal{L}\left\{R\frac{d\theta_R}{dt}\right\} \quad \rightarrow \quad V_\theta(R,s) = R[s\Theta_R(s) - \underbrace{\theta_R(0)}_{=0}]$$

$$v_\theta(aR,t) = aR\frac{d\theta_{aR}}{dt} \quad \rightarrow \quad \mathcal{L}\{v_\theta(aR,t)\} = \mathcal{L}\left\{aR\frac{d\theta_{aR}}{dt}\right\} \quad \rightarrow \quad V_\theta(aR,s) = aR\Omega_{aR}(s)$$

Note that since the cup's angular position $\theta_{aR}(t)$ is prescribed, $\Omega_{aR}(s)$ is a known function of s. Apply the transformed boundary conditions now to determine $C_1(s)$ and $C_2(s)$.

$$V_\theta(R,s) = C_1(s)I_1\left(\sqrt{\frac{\rho s}{\mu}}R\right) + C_2(s)K_1\left(\sqrt{\frac{\rho s}{\mu}}R\right) = Rs\Theta_R(s)$$

$$V_\theta(aR,s) = C_1(s)I_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right) + C_2(s)K_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right) = aR\Omega_{aR}(s)$$

Solving this system of equations yields

$$C_1(s) = R\frac{aK_1\left(\sqrt{\frac{\rho s}{\mu}}R\right)\Omega_{aR}(s) - sK_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)\Theta_R(s)}{I_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)K_1\left(\sqrt{\frac{\rho s}{\mu}}R\right) - I_1\left(\sqrt{\frac{\rho s}{\mu}}R\right)K_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)}$$

$$C_2(s) = -R\frac{aI_1\left(\sqrt{\frac{\rho s}{\mu}}R\right)\Omega_{aR}(s) - sI_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)\Theta_R(s)}{I_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)K_1\left(\sqrt{\frac{\rho s}{\mu}}R\right) - I_1\left(\sqrt{\frac{\rho s}{\mu}}R\right)K_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)}.$$

Now take the Laplace transform of both sides of the bob's equation of motion.

$$\mathcal{L}\left\{-k\theta_R + 2\pi R^3\mu L\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\bigg|_{r=R}\right\} = \mathcal{L}\left\{I\frac{d^2\theta_R}{dt^2}\right\}$$

Use the fact that the transform is linear.

$$-k\mathcal{L}\{\theta_R\} + 2\pi R^3\mu L\mathcal{L}\left\{\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\bigg|_{r=R}\right\} = I\mathcal{L}\left\{\frac{d^2\theta_R}{dt^2}\right\}$$

Transform the derivatives.

$$-k\Theta_R(s) + 2\pi R^3\mu L\frac{d}{dr}\left(\frac{V_\theta}{r}\right)\bigg|_{r=R} = I\left[s^2\Theta_R(s) - \underbrace{s\theta_R(0)}_{=0} - \underbrace{\frac{d\theta_R}{dt}(0)}_{=0}\right]$$

Bring the terms with $\Theta_R(s)$ to the right side.

$$2\pi R^3\mu L\frac{d}{dr}\left(\frac{V_\theta}{r}\right)\bigg|_{r=R} = (k + Is^2)\Theta_R(s)$$

Plug in the general solution for V_θ.

$$2\pi R^3\mu L\frac{d}{dr}\left[C_1(s)\frac{I_1\left(\sqrt{\frac{\rho s}{\mu}}r\right)}{r} + C_2(s)\frac{K_1\left(\sqrt{\frac{\rho s}{\mu}}r\right)}{r}\right]\bigg|_{r=R} = (k + Is^2)\Theta_R(s)$$

Evaluate the derivative.

$$2\pi R^3\mu L\left[C_1(s)\sqrt{\frac{\rho s}{\mu}}\frac{I_2\left(\sqrt{\frac{\rho s}{\mu}}r\right)}{r} - C_2(s)\sqrt{\frac{\rho s}{\mu}}\frac{K_2\left(\sqrt{\frac{\rho s}{\mu}}r\right)}{r}\right]\Bigg|_{r=R} = (k + Is^2)\Theta_R(s)$$

Note that I_2 and K_2 are the second-order modified Bessel functions of the first and second kind, respectively. Plug in $r = R$.

$$2\pi R^3\mu L\left[C_1(s)\sqrt{\frac{\rho s}{\mu}}\frac{I_2\left(\sqrt{\frac{\rho s}{\mu}}R\right)}{R} - C_2(s)\sqrt{\frac{\rho s}{\mu}}\frac{K_2\left(\sqrt{\frac{\rho s}{\mu}}R\right)}{R}\right] = (k + Is^2)\Theta_R(s)$$

Factor the left side.

$$2\pi R^2 L\sqrt{\mu\rho s}\left[C_1(s)I_2\left(\sqrt{\frac{\rho s}{\mu}}R\right) - C_2(s)K_2\left(\sqrt{\frac{\rho s}{\mu}}R\right)\right] = (k + Is^2)\Theta_R(s)$$

Divide both sides by $2\pi R^2 L\sqrt{\mu\rho s}$.

$$C_1(s)I_2\left(\sqrt{\frac{\rho s}{\mu}}R\right) - C_2(s)K_2\left(\sqrt{\frac{\rho s}{\mu}}R\right) = \frac{k + Is^2}{2\pi R^2 L\sqrt{\mu\rho s}}\Theta_R(s)$$

Substitute the formulas for $C_1(s)$ and $C_2(s)$.

$$R\frac{aK_1\left(\sqrt{\frac{\rho s}{\mu}}R\right)\Omega_{aR}(s) - sK_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)\Theta_R(s)}{I_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)K_1\left(\sqrt{\frac{\rho s}{\mu}}R\right) - I_1\left(\sqrt{\frac{\rho s}{\mu}}R\right)K_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)}I_2\left(\sqrt{\frac{\rho s}{\mu}}R\right)$$
$$+ R\frac{aI_1\left(\sqrt{\frac{\rho s}{\mu}}R\right)\Omega_{aR}(s) - sI_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)\Theta_R(s)}{I_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)K_1\left(\sqrt{\frac{\rho s}{\mu}}R\right) - I_1\left(\sqrt{\frac{\rho s}{\mu}}R\right)K_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)}K_2\left(\sqrt{\frac{\rho s}{\mu}}R\right) = \frac{k + Is^2}{2\pi R^2 L\sqrt{\mu\rho s}}\Theta_R(s)$$

Solve this equation for $\Theta_R(s)$.

$$\Theta_R(s) = \frac{2\pi R^2 a\mu L}{\left[2\pi R^3 L\sqrt{\mu\rho s^3}K_2\left(\sqrt{\frac{\rho s}{\mu}}R\right) + (k + Is^2)K_1\left(\sqrt{\frac{\rho s}{\mu}}R\right)\right]I_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right) + \left[2\pi R^3 L\sqrt{\mu\rho s^3}I_2\left(\sqrt{\frac{\rho s}{\mu}}R\right) - (k + Is^2)I_1\left(\sqrt{\frac{\rho s}{\mu}}R\right)\right]K_1\left(\sqrt{\frac{\rho s}{\mu}}aR\right)}\Omega_{aR}(s)$$

The final step is to take the inverse Laplace transform of $\Theta_R(s)$ to get $\theta_R(t)$. Since $\Theta_R(s)$ is a product of two functions, namely $\Omega_{aR}(s)$ and the big fraction in front of it [call this fraction $G_1(s)$], the convolution theorem can be applied to express $\theta_R(t)$ as a convolution integral.

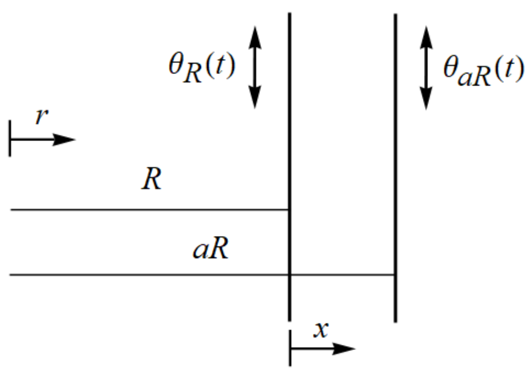
$$\begin{aligned}
\theta_R(t) &= \mathcal{L}^{-1}\{\Theta_R(s)\} \\
&= \mathcal{L}^{-1}\{G_1(s)\Omega_{aR}(s)\} \\
&= \int_0^t g_1(t-\xi)\frac{d\theta_{aR}}{d\xi}(\xi)\,d\xi \\
&= g_1(t-\xi)\theta_{aR}(\xi)\Big|_0^t - \int_0^t \frac{\partial}{\partial \xi}[g_1(t-\xi)]\theta_{aR}(\xi)\,d\xi \\
&= g_1(0)\theta_{aR}(t) - g_1(t)\theta_{aR}(0) - \int_0^t [-g_1'(t-\xi)]\theta_{aR}(\xi)\,d\xi
\end{aligned}$$

Therefore, assuming the inverse Laplace transform of the big fraction can be found,

$$\boxed{\theta_R(t) = g_1(0)\theta_{aR}(t) - g_1(t)\theta_{aR}(0) + \int_0^t g_1'(t-\xi)\theta_{aR}(\xi)\,d\xi.}$$

Part (d)

Since this exact solution is unfeasible, the assumption that the curvature is negligible will be made. The fluid now essentially flows between two parallel plates with length $2\pi R$ and width L.



To obtain the equations of motion for the bob and fluid, one could start over, using τ_{xz} in place of $\tau_{r\theta}$ and expanding the Navier-Stokes equation in Cartesian coordinates, but it's more instructive to simplify the equations already derived.

$$\text{(bob)} \qquad -k\theta_R + 2\pi R^3\mu L\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\Big|_{r=R} = I\frac{d^2\theta_R}{dt^2}$$

$$\text{(fluid)} \qquad \rho\frac{\partial v_\theta}{\partial t} = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)$$

r can be replaced in these equations with R to a good approximation because $R < r < aR$ and $a = 1 + \epsilon$, where $\epsilon \ll 1$.

$$\text{(bob)} \qquad -k\theta_R + 2\pi R^3\mu L\frac{\partial}{\partial r}\left(\frac{v_\theta}{R}\right)\Big|_{r=R} = I\frac{d^2\theta_R}{dt^2}$$

$$\text{(fluid)} \qquad \rho\frac{\partial v_\theta}{\partial t} = \mu\frac{\partial}{\partial r}\left(\frac{1}{R}\frac{\partial}{\partial r}(Rv_\theta)\right)$$

Simplify them.

$$\text{(bob)} \qquad -k\theta_R + 2\pi R^2\mu L\left.\frac{\partial v_\theta}{\partial r}\right|_{r=R} = I\frac{d^2\theta_R}{dt^2}$$

$$\text{(fluid)} \qquad \rho\frac{\partial v_\theta}{\partial t} = \mu\frac{\partial^2 v_\theta}{\partial r^2}$$

Define the dimensionless distance x to be

$$x = \frac{r-R}{aR-R}$$

so that $x=0$ at the left plate ($r=R$) and $x=1$ at the right plate ($r=aR$). Then, using the chain rule, the derivatives in terms of this new variable are

$$\frac{\partial}{\partial r} = \frac{dx}{dr}\frac{\partial}{\partial x} = \frac{1}{aR-R}\frac{\partial}{\partial x} \quad \Rightarrow \quad \frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right) = \frac{1}{aR-R}\frac{\partial}{\partial x}\left(\frac{1}{aR-R}\frac{\partial}{\partial x}\right) = \frac{1}{(aR-R)^2}\frac{\partial^2}{\partial x^2}.$$

Substitute these new derivatives into the equations of motion and replace $r=R$ with $x=0$.

$$\text{(bob)} \qquad -k\theta_R + 2\pi R^2\mu L\frac{1}{aR-R}\left.\frac{\partial v_\theta}{\partial x}\right|_{x=0} = I\frac{d^2\theta_R}{dt^2}$$

$$\text{(fluid)} \qquad \rho\frac{\partial v_\theta}{\partial t} = \mu\frac{1}{(aR-R)^2}\frac{\partial^2 v_\theta}{\partial x^2}$$

Simplify them.

$$\text{(bob)} \qquad -k\theta_R + \frac{2\pi R\mu L}{a-1}\left.\frac{\partial v_\theta}{\partial x}\right|_{x=0} = I\frac{d^2\theta_R}{dt^2}$$

$$\text{(fluid)} \qquad \rho\frac{\partial v_\theta}{\partial t} = \mu\frac{1}{(a-1)^2R^2}\frac{\partial^2 v_\theta}{\partial x^2}$$

Divide both sides of the first equation by k and divide both sides of the second equation by ρ.

$$\text{(bob)} \qquad -\theta_R + \frac{2\pi R\mu L}{(a-1)k}\left.\frac{\partial v_\theta}{\partial x}\right|_{x=0} = \frac{I}{k}\frac{d^2\theta_R}{dt^2}$$

$$\text{(fluid)} \qquad \frac{\partial v_\theta}{\partial t} = \frac{\mu/\rho}{(a-1)^2R^2}\frac{\partial^2 v_\theta}{\partial x^2}$$

Define the dimensionless time τ to be

$$\tau = \sqrt{\frac{k}{I}}t$$

so that

$$\frac{\partial}{\partial t} = \frac{d\tau}{dt}\frac{\partial}{\partial \tau} = \sqrt{\frac{k}{I}}\frac{\partial}{\partial \tau} \quad \Rightarrow \quad \frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \sqrt{\frac{k}{I}}\frac{d}{d\tau}\left(\sqrt{\frac{k}{I}}\frac{d}{d\tau}\right) = \frac{k}{I}\frac{d^2}{d\tau^2}.$$

Substitute these new derivatives into the equations of motion.

$$\text{(bob)} \qquad -\theta_R + \frac{2\pi R\mu L}{(a-1)k}\left.\frac{\partial v_\theta}{\partial x}\right|_{x=0} = \frac{I}{k}\frac{k}{I}\frac{d^2\theta_R}{d\tau^2}$$

$$\text{(fluid)} \qquad \sqrt{\frac{k}{I}}\frac{\partial v_\theta}{\partial \tau} = \frac{\mu/\rho}{(a-1)^2R^2}\frac{\partial^2 v_\theta}{\partial x^2}$$

Multiply both sides of the second equation by $\sqrt{I/k}$.

$$\text{(bob)} \qquad -\theta_R + \frac{2\pi R\mu L}{(a-1)k}\frac{\partial v_\theta}{\partial x}\bigg|_{x=0} = \frac{d^2\theta_R}{d\tau^2}$$

$$\text{(fluid)} \qquad \frac{\partial v_\theta}{\partial \tau} = \frac{\mu/\rho}{(a-1)^2R^2}\sqrt{\frac{I}{k}}\frac{\partial^2 v_\theta}{\partial x^2}$$

Define the dimensionless viscosity M to be

$$M = \frac{\mu/\rho}{(a-1)^2R^2}\sqrt{\frac{I}{k}}$$

and write the coefficient of $\partial v_\theta/\partial x|_{x=0}$ in terms of it.

$$\text{(bob)} \qquad -\theta_R + \frac{\mu/\rho}{(a-1)^2R^2}\sqrt{\frac{I}{k}}\frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}\frac{\partial v_\theta}{\partial x}\bigg|_{x=0} = \frac{d^2\theta_R}{d\tau^2}$$

$$\text{(fluid)} \qquad \frac{\partial v_\theta}{\partial \tau} = M\frac{\partial^2 v_\theta}{\partial x^2}$$

Bring the last fraction inside the derivative.

$$\text{(bob)} \qquad -\theta_R + \frac{\mu/\rho}{(a-1)^2R^2}\sqrt{\frac{I}{k}}\frac{\partial}{\partial x}\left[\frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}v_\theta\right]\bigg|_{x=0} = \frac{d^2\theta_R}{d\tau^2}$$

$$\text{(fluid)} \qquad \frac{\partial v_\theta}{\partial \tau} = M\frac{\partial^2 v_\theta}{\partial x^2}$$

Define the dimensionless velocity ϕ to be

$$\phi = \frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}v_\theta$$

and multiply both sides of the second equation by $[2\pi R^3 L\rho(a-1)]/\sqrt{kI}$.

$$\text{(bob)} \qquad -\theta_R + M\frac{\partial \phi}{\partial x}\bigg|_{x=0} = \frac{d^2\theta_R}{d\tau^2}$$

$$\text{(fluid)} \qquad \frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}\frac{\partial v_\theta}{\partial \tau} = M\frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}\frac{\partial^2 v_\theta}{\partial x^2}$$

Bring the constants except M inside the derivatives.

$$\text{(bob)} \qquad -\theta_R + M\frac{\partial \phi}{\partial x}\bigg|_{x=0} = \frac{d^2\theta_R}{d\tau^2}$$

$$\text{(fluid)} \qquad \frac{\partial}{\partial \tau}\left[\frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}v_\theta\right] = M\frac{\partial^2}{\partial x^2}\left[\frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}v_\theta\right]$$

Make the final substitution.

$$\text{(bob)} \qquad -\theta_R + M\frac{\partial \phi}{\partial x}\bigg|_{x=0} = \frac{d^2\theta_R}{d\tau^2}$$

$$\text{(fluid)} \qquad \frac{\partial \phi}{\partial \tau} = M\frac{\partial^2 \phi}{\partial x^2}$$

Now transform the initial conditions.

$$\theta_R(t=0)=0 \quad \rightarrow \quad \theta_R\left(\sqrt{\frac{I}{k}}\tau=0\right)=0 \quad \rightarrow \quad \theta_R(\tau=0)=0$$

$$\frac{d\theta_R}{dt}(t=0)=0 \quad \rightarrow \quad \sqrt{\frac{k}{I}}\frac{d\theta_R}{d\tau}\left(\sqrt{\frac{I}{k}}\tau=0\right)=0 \quad \rightarrow \quad \frac{d\theta_R}{d\tau}(\tau=0)=0$$

$$v_\theta(r,t=0)=0 \quad \rightarrow \quad \frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}v_\theta\left(x,\sqrt{\frac{I}{k}}\tau=0\right)=0 \quad \rightarrow \quad \phi(x,\tau=0)=0$$

Finally, transform the boundary conditions.

$$v_\theta(r=R,t)=R\frac{d\theta_R}{dt} \quad \rightarrow \quad \frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}v_\theta(x=0,\tau)=\frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}R\sqrt{\frac{k}{I}}\frac{d\theta_R}{d\tau}$$

$$v_\theta(r=aR,t)=aR\frac{d\theta_{aR}}{dt} \quad \rightarrow \quad \frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}v_\theta(x=1,\tau)=\frac{2\pi R^3 L\rho(a-1)}{\sqrt{kI}}aR\sqrt{\frac{k}{I}}\frac{d\theta_{aR}}{d\tau}$$

Define the dimensionless reciprocal of moment of inertia to be

$$A=\frac{2\pi R^4 L\rho(a-1)}{I}$$

and substitute ϕ into each left side.

$$\phi(x=0,\tau)=\frac{2\pi R^4 L\rho(a-1)}{I}\frac{d\theta_R}{dt}=A\frac{d\theta_R}{d\tau}$$

$$\phi(x=1,\tau)=\frac{2\pi R^4 L\rho(a-1)}{I}a\frac{d\theta_R}{dt}=Aa\frac{d\theta_{aR}}{d\tau}\approx A\frac{d\theta_{aR}}{d\tau}$$

Therefore, under the additional assumption of negligible curvature, the nondimensionalized equations of motion for the bob and fluid are

$$\text{(bob)} \qquad \frac{d^2\theta_R}{d\tau^2}=-\theta_R+M\left(\frac{\partial\phi}{\partial x}\right)\bigg|_{x=0} \qquad \begin{cases}\theta_R(0)=0\\ \dfrac{d\theta_R}{d\tau}(0)=0\end{cases}$$

$$\text{(fluid)} \qquad \frac{\partial\phi}{\partial\tau}=M\frac{\partial^2\phi}{\partial x^2} \qquad \begin{cases}\phi(x,0)=0\\ \phi(0,\tau)=A\dfrac{d\theta_R}{d\tau}\\ \phi(1,\tau)=A\dfrac{d\theta_{aR}}{d\tau}\end{cases},$$

where M, A, and $\theta_{aR}(\tau)$ are known. Since both equations and their associated conditions are linear, the Laplace transform can be applied to solve for $\theta_R(\tau)$ and $\phi(x,\tau)$. Take the Laplace transform of both sides of the fluid's equation of motion.

$$\mathcal{L}\left\{\frac{\partial\phi}{\partial\tau}\right\}=\mathcal{L}\left\{M\frac{\partial^2\phi}{\partial x^2}\right\}$$

Use the fact that the transform is linear.

$$\mathcal{L}\left\{\frac{\partial\phi}{\partial\tau}\right\}=M\mathcal{L}\left\{\frac{\partial^2\phi}{\partial x^2}\right\}$$

Transform the derivatives.

$$s\Phi(x,s) - \underbrace{\phi(x,0)}_{=0} = M\frac{d^2\Phi}{dx^2}$$

Divide both sides by M.

$$\frac{d^2\Phi}{dx^2} = \frac{s}{M}\Phi(x,s)$$

The general solution to this ODE can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Phi(x,s) = C_3(s)\cosh\left(\sqrt{\frac{s}{M}}x\right) + C_4(s)\sinh\left(\sqrt{\frac{s}{M}}x\right)$$

In order to determine $C_3(s)$ and $C_4(s)$, the boundary conditions for ϕ need to be transformed.

$$\phi(0,\tau) = A\frac{d\theta_R}{d\tau} \quad \rightarrow \quad \mathcal{L}\{\phi(0,\tau)\} = \mathcal{L}\left\{A\frac{d\theta_R}{d\tau}\right\} \quad \rightarrow \quad \Phi(0,s) = A[s\Theta_R(s) - \underbrace{\theta_R(0)}_{=0}]$$

$$\phi(1,\tau) = A\frac{d\theta_{aR}}{d\tau} \quad \rightarrow \quad \mathcal{L}\{\phi(1,\tau)\} = \mathcal{L}\left\{A\frac{d\theta_{aR}}{d\tau}\right\} \quad \rightarrow \quad \Phi(1,s) = A\Omega_{aR}(s)$$

Apply the transformed boundary conditions now.

$$\Phi(0,s) = C_3(s) = As\Theta_R(s)$$

$$\Phi(1,s) = C_3(s)\cosh\left(\sqrt{\frac{s}{M}}\right) + C_4(s)\sinh\left(\sqrt{\frac{s}{M}}\right) = A\Omega_{aR}(s)$$

Solve this system of equations for $C_3(s)$ and $C_4(s)$.

$$C_3(s) = As\Theta_R(s)$$

$$C_4(s) = A\Omega_{aR}(s)\frac{1}{\sinh\left(\sqrt{\frac{s}{M}}\right)} - As\Theta_R(s)\frac{\cosh\left(\sqrt{\frac{s}{M}}\right)}{\sinh\left(\sqrt{\frac{s}{M}}\right)}$$

Now take the Laplace transform of both sides of the bob's equation of motion.

$$\mathcal{L}\left\{\frac{d^2\theta_R}{d\tau^2}\right\} = \mathcal{L}\left\{-\theta_R + M\left(\frac{\partial\phi}{\partial x}\right)\bigg|_{x=0}\right\}$$

Use the fact that the transform is linear.

$$\mathcal{L}\left\{\frac{d^2\theta_R}{d\tau^2}\right\} = -\mathcal{L}\{\theta_R\} + M\mathcal{L}\left\{\left(\frac{\partial\phi}{\partial x}\right)\bigg|_{x=0}\right\}$$

Transform the derivatives.

$$s^2\Theta_R(s) - \underbrace{s\theta_R(0)}_{=0} - \underbrace{\frac{d\theta_R}{d\tau}(0)}_{=0} = -\Theta_R(s) + M\left(\frac{d\Phi}{dx}\right)\bigg|_{x=0}$$

Bring the terms with $\Theta_R(s)$ to the left side and substitute the general solution for $\Phi(x,s)$.

$$(s^2+1)\Theta_R(s) = M\frac{d}{dx}\left[C_3(s)\cosh\left(\sqrt{\frac{s}{M}}x\right) + C_4(s)\sinh\left(\sqrt{\frac{s}{M}}x\right)\right]\bigg|_{x=0}$$

Evaluate the derivative.

$$(s^2+1)\Theta_R(s) = M\left[C_3(s)\sqrt{\frac{s}{M}}\sinh\left(\sqrt{\frac{s}{M}}x\right) + C_4(s)\sqrt{\frac{s}{M}}\cosh\left(\sqrt{\frac{s}{M}}x\right)\right]\Bigg|_{x=0}$$

Set $x = 0$.

$$(s^2+1)\Theta_R(s) = M\left[C_4(s)\sqrt{\frac{s}{M}}\right]$$

Substitute the formula found for $C_4(s)$.

$$(s^2+1)\Theta_R(s) = \sqrt{Ms}\left[A\Omega_{aR}(s)\frac{1}{\sinh\left(\sqrt{\frac{s}{M}}\right)} - As\Theta_R(s)\frac{\cosh\left(\sqrt{\frac{s}{M}}\right)}{\sinh\left(\sqrt{\frac{s}{M}}\right)}\right]$$

Solve this equation for $\Theta_R(s)$.

$$\Theta_R(s) = \frac{A\sqrt{Ms}}{(s^2+1)\sinh\sqrt{\frac{s}{M}} + A\sqrt{Ms^3}\cosh\sqrt{\frac{s}{M}}}\Omega_{aR}(s)$$

The final step is to take the inverse Laplace transform of $\Theta_R(s)$ to get $\theta_R(t)$. Since $\Theta_R(s)$ is a product of two functions, namely $\Omega_{aR}(s)$ and the small fraction in front of it [call this fraction $G_2(s)$], the convolution theorem can be applied to express $\theta_R(t)$ as a convolution integral.

$$\begin{aligned}
\theta_R(t) &= \mathcal{L}^{-1}\{\Theta_R(s)\} \\
&= \mathcal{L}^{-1}\{G_2(s)\Omega_{aR}(s)\} \\
&= \int_0^t g_2(t-\xi)\frac{d\theta_{aR}}{d\xi}(\xi)\,d\xi \\
&= g_2(t-\xi)\theta_{aR}(\xi)\Big|_0^t - \int_0^t \frac{\partial}{\partial\xi}[g_2(t-\xi)]\theta_{aR}(\xi)\,d\xi \\
&= g_2(0)\theta_{aR}(t) - g_2(t)\theta_{aR}(0) - \int_0^t [-g_2'(t-\xi)]\theta_{aR}(\xi)\,d\xi
\end{aligned}$$

Therefore, assuming the inverse Laplace transform of the small fraction can be found,

$$\boxed{\theta_R(t) = g_2(0)\theta_{aR}(t) - g_2(t)\theta_{aR}(0) + \int_0^t g_2'(t-\xi)\theta_{aR}(\xi)\,d\xi.}$$

Part (e)

Since the attempts to solve the problem exactly have not yielded any formulas of practical use, a steady-state solution will be settled for. Knowing that the cup has sinusoidal oscillations, we can postulate that

$$\theta_{aR}(\tau) = C_5 \cos \overline{\omega}\tau,$$

where $\overline{\omega} = \omega/\omega_0 = \omega\sqrt{I/k}$ is some dimensionless frequency (made so because τ is dimensionless). At the steady state the bob and fluid are assumed to oscillate at this same frequency but have different amplitudes and phases.

$$\theta_R(\tau) = C_6 \cos(\overline{\omega}\tau + \alpha)$$
$$\phi(x,\tau) = C_7(x)\cos(\overline{\omega}\tau + \psi)$$

Because of these phases, the amplitudes of θ_R and ϕ become complex.

$$\theta_R(\tau) = \Re\{C_6 e^{i(\overline{\omega}\tau+\alpha)}\} = \Re\{C_6 e^{i\alpha} e^{i\overline{\omega}\tau}\} = \Re\{\overset{\circ}{\theta}_R e^{i\overline{\omega}\tau}\}$$
$$\phi(x,\tau) = \Re\{C_7(x) e^{i(\overline{\omega}\tau+\psi)}\} = \Re\{C_7(x) e^{i\psi} e^{i\overline{\omega}\tau}\} = \Re\{\phi^\circ(x) e^{i\overline{\omega}\tau}\}$$

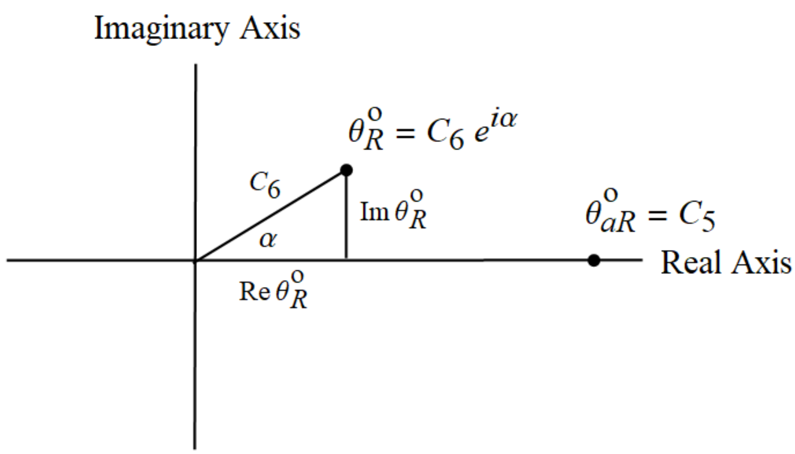
By comparing this formula for $\theta_R(\tau)$ to

$$\theta_{aR}(\tau) = \Re\{C_5 e^{i\overline{\omega}\tau}\} = \Re\{\overset{\circ}{\theta}_{aR} e^{i\overline{\omega}\tau}\},$$

we see the output-to-input amplitude ratio is

$$\frac{|\overset{\circ}{\theta}_R|}{|\overset{\circ}{\theta}_{aR}|} = \frac{|\overset{\circ}{\theta}_R|}{\overset{\circ}{\theta}_{aR}}.$$

Geometrically, $|\overset{\circ}{\theta}_R|$ and $|\overset{\circ}{\theta}_{aR}|$ are the respective distances from the origin to the points, $\overset{\circ}{\theta}_R$ and $\overset{\circ}{\theta}_{aR}$, in the complex plane.



Note that

$$\overset{\circ}{\theta}_R = \Re\{\overset{\circ}{\theta}_R\} + i\Im\{\overset{\circ}{\theta}_R\}$$

so the phase angle α is given by

$$\tan\alpha = \frac{\Im\{\overset{\circ}{\theta}_R\}}{\Re\{\overset{\circ}{\theta}_R\}}.$$

Part (f)

Substitute the postulated steady-state formulas,

$$\theta_{aR}(\tau) = \Re\{\overset{\circ}{\theta}_{aR} e^{i\varpi\tau}\}$$
$$\theta_R(\tau) = \Re\{\overset{\circ}{\theta}_R e^{i\varpi\tau}\}$$
$$\phi(x,\tau) = \Re\{\phi^\circ(x) e^{i\varpi\tau}\},$$

into the nondimensionalized equations of part (d).

$$\text{(bob)} \qquad \frac{d^2\theta_R}{d\tau^2} = -\theta_R + M\left(\frac{\partial\phi}{\partial x}\right)\bigg|_{x=0} \qquad \begin{cases} \theta_R(0) = 0 \\ \dfrac{d\theta_R}{d\tau}(0) = 0 \end{cases}$$

$$\text{(fluid)} \qquad \frac{\partial\phi}{\partial\tau} = M\frac{\partial^2\phi}{\partial x^2} \qquad \begin{cases} \phi(x,0) = 0 \\ \phi(0,\tau) = A\dfrac{d\theta_R}{d\tau} \\ \phi(1,\tau) = A\dfrac{d\theta_{aR}}{d\tau} \end{cases}$$

Since we're interested in the steady state, the initial conditions are irrelevant.

$$\text{(bob)} \qquad \frac{d^2}{d\tau^2}\Re\{\overset{\circ}{\theta}_R e^{i\varpi\tau}\} = -\Re\{\overset{\circ}{\theta}_R e^{i\varpi\tau}\} + M\left(\frac{\partial}{\partial x}\Re\{\phi^\circ(x)e^{i\varpi\tau}\}\right)\bigg|_{x=0}$$

$$\text{(fluid)} \qquad \frac{\partial}{\partial\tau}\Re\{\phi^\circ(x)e^{i\varpi\tau}\} = M\frac{\partial^2}{\partial x^2}\Re\{\phi^\circ(x)e^{i\varpi\tau}\} \qquad \begin{cases} \Re\{\phi^\circ(0)e^{i\varpi\tau}\} = A\dfrac{d}{d\tau}\Re\{\overset{\circ}{\theta}_R e^{i\varpi\tau}\} \\ \Re\{\phi^\circ(1)e^{i\varpi\tau}\} = A\dfrac{d}{d\tau}\Re\{\overset{\circ}{\theta}_{aR} e^{i\varpi\tau}\} \end{cases}$$

Evaluate the derivatives.

$$\text{(bob)} \qquad \Re\{\overset{\circ}{\theta}_R (i\varpi)^2 e^{i\varpi\tau}\} = -\Re\{\overset{\circ}{\theta}_R e^{i\varpi\tau}\} + M\Re\left\{\frac{d\phi^\circ}{dx}\bigg|_{x=0} e^{i\varpi\tau}\right\}$$

$$\text{(fluid)} \qquad \Re\{(i\varpi)\phi^\circ(x)e^{i\varpi\tau}\} = M\Re\left\{\frac{d^2\phi^\circ}{dx^2}e^{i\varpi\tau}\right\} \qquad \begin{cases} \Re\{\phi^\circ(0)e^{i\varpi\tau}\} = A\Re\{\overset{\circ}{\theta}_R(i\varpi)e^{i\varpi\tau}\} \\ \Re\{\phi^\circ(1)e^{i\varpi\tau}\} = A\Re\{\overset{\circ}{\theta}_{aR}(i\varpi)e^{i\varpi\tau}\} \end{cases}$$

Since A and M are real, the real operator is effectively linear.

$$\text{(bob)} \qquad \Re\{\overset{\circ}{\theta}_R (i\varpi)^2 e^{i\varpi\tau}\} = \Re\left\{-\overset{\circ}{\theta}_R e^{i\varpi\tau} + M\frac{d\phi^\circ}{dx}\bigg|_{x=0} e^{i\varpi\tau}\right\}$$

$$\text{(fluid)} \qquad \Re\{(i\varpi)\phi^\circ(x)e^{i\varpi\tau}\} = \Re\left\{M\frac{d^2\phi^\circ}{dx^2}e^{i\varpi\tau}\right\} \qquad \begin{cases} \Re\{\phi^\circ(0)e^{i\varpi\tau}\} = \Re\{A\overset{\circ}{\theta}_R(i\varpi)e^{i\varpi\tau}\} \\ \Re\{\phi^\circ(1)e^{i\varpi\tau}\} = \Re\{A\overset{\circ}{\theta}_{aR}(i\varpi)e^{i\varpi\tau}\} \end{cases}$$

Remove the real operators.

$$\text{(bob)} \qquad \overset{\circ}{\theta}_R (i\varpi)^2 e^{i\varpi\tau} = -\overset{\circ}{\theta}_R e^{i\varpi\tau} + M\frac{d\phi^\circ}{dx}\bigg|_{x=0} e^{i\varpi\tau}$$

$$\text{(fluid)} \qquad (i\varpi)\phi^\circ(x)e^{i\varpi\tau} = M\frac{d^2\phi^\circ}{dx^2}e^{i\varpi\tau} \qquad \begin{cases} \phi^\circ(0)e^{i\varpi\tau} = A\overset{\circ}{\theta}_R(i\varpi)e^{i\varpi\tau} \\ \phi^\circ(1)e^{i\varpi\tau} = A\overset{\circ}{\theta}_{aR}(i\varpi)e^{i\varpi\tau} \end{cases}$$

Therefore, dividing both sides of each equation by $e^{i\varpi\tau}$,

(bob) $$-\varpi^2\overset{\circ}{\theta}_R = -\overset{\circ}{\theta}_R + M\left.\frac{d\phi^\circ}{dx}\right|_{x=0}$$

(fluid) $$i\varpi\phi^\circ = M\frac{d^2\phi^\circ}{dx^2} \qquad \begin{cases}\phi^\circ(0) = iA\varpi\overset{\circ}{\theta}_R \\ \phi^\circ(1) = iA\varpi\overset{\circ}{\theta}_{aR}\end{cases}.$$

Part (g)

Begin by solving the fluid's equation of motion.

$$\frac{d^2\phi^\circ}{dx^2} = \frac{i\varpi}{M}\phi^\circ$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi^\circ(x) = C_8\cosh\left(\sqrt{\frac{i\varpi}{M}}x\right) + C_9\sinh\left(\sqrt{\frac{i\varpi}{M}}x\right)$$

Apply the boundary conditions to determine C_8 and C_9.

$$\phi^\circ(0) = C_8 = iA\varpi\overset{\circ}{\theta}_R$$
$$\phi^\circ(1) = C_8\cosh\sqrt{\frac{i\varpi}{M}} + C_9\sinh\sqrt{\frac{i\varpi}{M}} = iA\varpi\overset{\circ}{\theta}_{aR}$$

Solving this system of equations yields

$$C_8 = iA\varpi\overset{\circ}{\theta}_R$$
$$C_9 = iA\varpi\left(\overset{\circ}{\theta}_{aR} - \overset{\circ}{\theta}_R\cosh\sqrt{\frac{i\varpi}{M}}\right)\frac{1}{\sinh\sqrt{\frac{i\varpi}{M}}}.$$

Therefore,

$$\phi^\circ(x) = iA\varpi\overset{\circ}{\theta}_R\cosh\left(\sqrt{\frac{i\varpi}{M}}x\right) + iA\varpi\left(\overset{\circ}{\theta}_{aR} - \overset{\circ}{\theta}_R\cosh\sqrt{\frac{i\varpi}{M}}\right)\frac{\sinh\left(\sqrt{\frac{i\varpi}{M}}x\right)}{\sinh\sqrt{\frac{i\varpi}{M}}}.$$

Now take a derivative of it with respect to x

$$\frac{d\phi^\circ}{dx} = iA\varpi\overset{\circ}{\theta}_R\sqrt{\frac{i\varpi}{M}}\sinh\left(\sqrt{\frac{i\varpi}{M}}x\right) + iA\varpi\left(\overset{\circ}{\theta}_{aR} - \overset{\circ}{\theta}_R\cosh\sqrt{\frac{i\varpi}{M}}\right)\sqrt{\frac{i\varpi}{M}}\frac{\cosh\left(\sqrt{\frac{i\varpi}{M}}x\right)}{\sinh\sqrt{\frac{i\varpi}{M}}}$$

and evaluate it at $x = 0$.

$$\left.\frac{d\phi^\circ}{dx}\right|_{x=0} = iA\varpi\left(\overset{\circ}{\theta}_{aR} - \overset{\circ}{\theta}_R\cosh\sqrt{\frac{i\varpi}{M}}\right)\sqrt{\frac{i\varpi}{M}}\frac{1}{\sinh\sqrt{\frac{i\varpi}{M}}}$$
$$= -\frac{A(i\varpi)^{3/2}}{\sqrt{M}}\left(\frac{\overset{\circ}{\theta}_R\cosh\sqrt{i\varpi/M} - \overset{\circ}{\theta}_{aR}}{\sinh\sqrt{i\varpi/M}}\right)$$

Part (h)

Substitute this result for $d\phi^\circ/dx|_{x=0}$ into the bob's equation of motion.

$$-\varpi^2\theta_R^\circ = -\theta_R^\circ + M\frac{d\phi^\circ}{dx}\bigg|_{x=0}$$

Doing so results in an algebraic equation for θ_R°.

$$-\varpi^2\theta_R^\circ = -\theta_R^\circ - M\frac{A(i\varpi)^{3/2}}{\sqrt{M}}\left(\frac{\theta_R^\circ\cosh\sqrt{i\varpi/M} - \theta_{aR}^\circ}{\sinh\sqrt{i\varpi/M}}\right)$$

Solve it for θ_R°.

$$\varpi^2\theta_R^\circ = \theta_R^\circ + iA\sqrt{iM\varpi^3}(\theta_R^\circ\coth\sqrt{i\varpi/M} - \theta_{aR}^\circ\operatorname{csch}\sqrt{i\varpi/M})$$

$$\varpi^2\theta_R^\circ = \theta_R^\circ + iA\sqrt{iM\varpi^3}\theta_R^\circ\coth\sqrt{i\varpi/M} - iA\sqrt{iM\varpi^3}\theta_{aR}^\circ\operatorname{csch}\sqrt{i\varpi/M}$$

$$(\varpi^2 - 1 - iA\sqrt{iM\varpi^3}\coth\sqrt{i\varpi/M})\theta_R^\circ = -iA\sqrt{iM\varpi^3}\theta_{aR}^\circ\operatorname{csch}\sqrt{i\varpi/M}$$

$$\begin{aligned}
\theta_R^\circ &= \theta_{aR}^\circ\frac{-iA\sqrt{iM\varpi^3}\operatorname{csch}\sqrt{i\varpi/M}}{\varpi^2 - 1 - iA\sqrt{iM\varpi^3}\coth\sqrt{i\varpi/M}}\cdot\frac{i}{i} \\
&= \theta_{aR}^\circ\frac{A\sqrt{iM\varpi^3}\operatorname{csch}\sqrt{i\varpi/M}}{i(\varpi^2-1) + A\sqrt{iM\varpi^3}\coth\sqrt{i\varpi/M}}\cdot\frac{\sqrt{\frac{iM}{\varpi}}}{\sqrt{\frac{iM}{\varpi}}} \\
&= \theta_{aR}^\circ\frac{iAM\varpi}{\sinh\sqrt{i\varpi/M}\left[i\sqrt{\frac{iM}{\varpi}}(\varpi^2-1) + iAM\varpi\coth\sqrt{i\varpi/M}\right]} \\
&= \theta_{aR}^\circ\frac{iAM\varpi}{i^2\sqrt{\frac{M}{i\varpi}}(\varpi^2-1)\sinh\sqrt{i\varpi/M} + iAM\varpi\cosh\sqrt{i\varpi/M}} \\
&= \theta_{aR}^\circ\frac{iAM\varpi}{-(\varpi^2-1)\frac{\sinh\sqrt{i\varpi/M}}{\sqrt{i\varpi/M}} + iAM\varpi\cosh\sqrt{i\varpi/M}}
\end{aligned}$$

Therefore, dividing both sides by θ_{aR}°,

$$\frac{\theta_R^\circ}{\theta_{aR}^\circ} = \frac{AMi\varpi}{(1-\varpi^2)\frac{\sinh\sqrt{i\varpi/M}}{\sqrt{i\varpi/M}} + AMi\varpi\cosh\sqrt{i\varpi/M}}. \tag{4C.2-17}$$

The aim now is to write the right side in rectangular form, $E_1(\varpi) + iE_2(\varpi)$, where E_1 and E_2 are functions to be determined. The amplitude ratio will then be

$$\left|\frac{\theta_R^\circ}{\theta_{aR}^\circ}\right| = |E_1(\varpi) + iE_2(\varpi)| = \sqrt{[E_1(\varpi)]^2 + [E_2(\varpi)]^2},$$

and the phase shift α will be

$$\tan\alpha = \frac{E_2(\varpi)}{E_1(\varpi)}.$$

Begin the simplification.

$$\begin{aligned}
\frac{\theta_R^\circ}{\theta_{aR}^\circ} &= \frac{AMi\varpi}{(1-\varpi^2)\frac{\sinh\sqrt{i\varpi/M}}{\sqrt{i\varpi/M}} + AMi\varpi\cosh\sqrt{i\varpi/M}} \cdot \frac{\sqrt{\frac{i\varpi}{M}}}{\sqrt{\frac{i\varpi}{M}}} \\
&= \frac{i^{3/2}A\sqrt{M\varpi^3}}{(1-\varpi^2)\sinh\sqrt{i\varpi/M} + iA\sqrt{M\varpi^3}i^{1/2}\cosh\sqrt{i\varpi/M}}
\end{aligned}$$

Note that

$$\begin{aligned}
i^{1/2} &= (e^{i\pi/2})^{1/2} = \pm e^{i\pi/4} \\
i^{3/2} &= (e^{i\pi/2})^{3/2} = \pm e^{3i\pi/4}.
\end{aligned}$$

As a result,

$$\frac{\theta_R^\circ}{\theta_{aR}^\circ} = \frac{\pm e^{3i\pi/4}A\sqrt{M\varpi^3}}{(1-\varpi^2)\sinh\sqrt{i\varpi/M} \pm iA\sqrt{M\varpi^3}e^{i\pi/4}\cosh\sqrt{i\varpi/M}}.$$

Recall that hyperbolic sine is defined as

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

so

$$\begin{aligned}
\sinh\sqrt{\frac{i\varpi}{M}} &= \frac{1}{2}\left[\exp\left(\sqrt{\frac{i\varpi}{M}}\right) - \exp\left(-\sqrt{\frac{i\varpi}{M}}\right)\right] \\
&= \frac{1}{2}\left[\exp\left(\pm\sqrt{\frac{\varpi}{M}}e^{i\pi/4}\right) - \exp\left(\mp\sqrt{\frac{\varpi}{M}}e^{i\pi/4}\right)\right] \\
&= \pm\frac{1}{2}\left[\exp\left(\sqrt{\frac{\varpi}{M}}e^{i\pi/4}\right) - \exp\left(-\sqrt{\frac{\varpi}{M}}e^{i\pi/4}\right)\right] \\
&= \pm\frac{1}{2}\left\{\exp\left[\sqrt{\frac{\varpi}{M}}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right] - \exp\left[-\sqrt{\frac{\varpi}{M}}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right]\right\} \\
&= \pm\frac{1}{2}\left[\exp\left(\sqrt{\frac{\varpi}{2M}} + i\sqrt{\frac{\varpi}{2M}}\right) - \exp\left(-\sqrt{\frac{\varpi}{2M}} - i\sqrt{\frac{\varpi}{2M}}\right)\right] \\
&= \pm\frac{1}{2}\left[\exp\left(\sqrt{\frac{\varpi}{2M}}\right)\exp\left(i\sqrt{\frac{\varpi}{2M}}\right) - \exp\left(-\sqrt{\frac{\varpi}{2M}}\right)\exp\left(-i\sqrt{\frac{\varpi}{2M}}\right)\right] \\
&= \pm\frac{1}{2}\left[\exp\left(\sqrt{\frac{\varpi}{2M}}\right)\left(\cos\sqrt{\frac{\varpi}{2M}} + i\sin\sqrt{\frac{\varpi}{2M}}\right) - \exp\left(-\sqrt{\frac{\varpi}{2M}}\right)\left(\cos\sqrt{\frac{\varpi}{2M}} - i\sin\sqrt{\frac{\varpi}{2M}}\right)\right] \\
&= \pm\frac{1}{2}\left\{\left[\exp\left(\sqrt{\frac{\varpi}{2M}}\right) - \exp\left(-\sqrt{\frac{\varpi}{2M}}\right)\right]\cos\sqrt{\frac{\varpi}{2M}}\right. \\
&\qquad\left. + i\left[\exp\left(\sqrt{\frac{\varpi}{2M}}\right) + \exp\left(-\sqrt{\frac{\varpi}{2M}}\right)\right]\sin\sqrt{\frac{\varpi}{2M}}\right\}.
\end{aligned}$$

Finish simplifying the right side.

$$\begin{aligned}\sinh\sqrt{\frac{i\overline{\omega}}{M}} &= \pm\frac{1}{2}\left[\left(2\sinh\sqrt{\frac{\overline{\omega}}{2M}}\right)\cos\sqrt{\frac{\overline{\omega}}{2M}} + i\left(2\cosh\sqrt{\frac{\overline{\omega}}{2M}}\right)\sin\sqrt{\frac{\overline{\omega}}{2M}}\right] \\ &= \pm\left(\sinh\sqrt{\frac{\overline{\omega}}{2M}}\cos\sqrt{\frac{\overline{\omega}}{2M}} + i\cosh\sqrt{\frac{\overline{\omega}}{2M}}\sin\sqrt{\frac{\overline{\omega}}{2M}}\right)\end{aligned}$$

Also, recall that hyperbolic cosine is defined as

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

so

$$\begin{aligned}\cosh\sqrt{\frac{i\overline{\omega}}{M}} &= \frac{1}{2}\left[\exp\left(\sqrt{\frac{i\overline{\omega}}{M}}\right) + \exp\left(-\sqrt{\frac{i\overline{\omega}}{M}}\right)\right] \\ &= \frac{1}{2}\left[\exp\left(\pm\sqrt{\frac{\overline{\omega}}{M}}e^{i\pi/4}\right) + \exp\left(\mp\sqrt{\frac{\overline{\omega}}{M}}e^{i\pi/4}\right)\right] \\ &= \frac{1}{2}\left[\exp\left(\sqrt{\frac{\overline{\omega}}{M}}e^{i\pi/4}\right) + \exp\left(-\sqrt{\frac{\overline{\omega}}{M}}e^{i\pi/4}\right)\right] \\ &= \frac{1}{2}\left\{\exp\left[\sqrt{\frac{\overline{\omega}}{M}}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right] + \exp\left[-\sqrt{\frac{\overline{\omega}}{M}}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right]\right\} \\ &= \frac{1}{2}\left[\exp\left(\sqrt{\frac{\overline{\omega}}{2M}} + i\sqrt{\frac{\overline{\omega}}{2M}}\right) + \exp\left(-\sqrt{\frac{\overline{\omega}}{2M}} - i\sqrt{\frac{\overline{\omega}}{2M}}\right)\right] \\ &= \frac{1}{2}\left[\exp\left(\sqrt{\frac{\overline{\omega}}{2M}}\right)\exp\left(i\sqrt{\frac{\overline{\omega}}{2M}}\right) + \exp\left(-\sqrt{\frac{\overline{\omega}}{2M}}\right)\exp\left(-i\sqrt{\frac{\overline{\omega}}{2M}}\right)\right] \\ &= \frac{1}{2}\left[\exp\left(\sqrt{\frac{\overline{\omega}}{2M}}\right)\left(\cos\sqrt{\frac{\overline{\omega}}{2M}} + i\sin\sqrt{\frac{\overline{\omega}}{2M}}\right) + \exp\left(-\sqrt{\frac{\overline{\omega}}{2M}}\right)\left(\cos\sqrt{\frac{\overline{\omega}}{2M}} - i\sin\sqrt{\frac{\overline{\omega}}{2M}}\right)\right] \\ &= \frac{1}{2}\left\{\left[\exp\left(\sqrt{\frac{\overline{\omega}}{2M}}\right) + \exp\left(-\sqrt{\frac{\overline{\omega}}{2M}}\right)\right]\cos\sqrt{\frac{\overline{\omega}}{2M}}\right. \\ &\qquad\left. + i\left[\exp\left(\sqrt{\frac{\overline{\omega}}{2M}}\right) - \exp\left(-\sqrt{\frac{\overline{\omega}}{2M}}\right)\right]\sin\sqrt{\frac{\overline{\omega}}{2M}}\right\} \\ &= \frac{1}{2}\left[\left(2\cosh\sqrt{\frac{\overline{\omega}}{2M}}\right)\cos\sqrt{\frac{\overline{\omega}}{2M}} + i\left(2\sinh\sqrt{\frac{\overline{\omega}}{2M}}\right)\sin\sqrt{\frac{\overline{\omega}}{2M}}\right] \\ &= \cosh\sqrt{\frac{\overline{\omega}}{2M}}\cos\sqrt{\frac{\overline{\omega}}{2M}} + i\sinh\sqrt{\frac{\overline{\omega}}{2M}}\sin\sqrt{\frac{\overline{\omega}}{2M}}.\end{aligned}$$

Substitute these results into the ratio $\theta^\circ_R/\theta^\circ_{aR}$ and continue to simplify it.

$$\begin{aligned}
\frac{\theta^\circ_R}{\theta^\circ_{aR}} &= \frac{\pm e^{3i\pi/4}A\sqrt{M\varpi^3}}{(1-\varpi^2)\sinh\sqrt{i\varpi/M} \pm iA\sqrt{M\varpi^3}e^{i\pi/4}\cosh\sqrt{i\varpi/M}} \\
&= \frac{\pm\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)A\sqrt{M\varpi^3}}{\pm(1-\varpi^2)\left(\sinh\sqrt{\frac{\varpi}{2M}}\cos\sqrt{\frac{\varpi}{2M}}+i\cosh\sqrt{\frac{\varpi}{2M}}\sin\sqrt{\frac{\varpi}{2M}}\right) \pm iA\sqrt{M\varpi^3}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\left(\cosh\sqrt{\frac{\varpi}{2M}}\cos\sqrt{\frac{\varpi}{2M}}+i\sinh\sqrt{\frac{\varpi}{2M}}\sin\sqrt{\frac{\varpi}{2M}}\right)} \\
&= \frac{\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)A\sqrt{M\varpi^3}}{(1-\varpi^2)\left(\sinh\sqrt{\frac{\varpi}{2M}}\cos\sqrt{\frac{\varpi}{2M}}+i\cosh\sqrt{\frac{\varpi}{2M}}\sin\sqrt{\frac{\varpi}{2M}}\right) + A\sqrt{M\varpi^3}\left(i\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)\left(\cosh\sqrt{\frac{\varpi}{2M}}\cos\sqrt{\frac{\varpi}{2M}}+i\sinh\sqrt{\frac{\varpi}{2M}}\sin\sqrt{\frac{\varpi}{2M}}\right)} \\
&= \frac{(-1+i)A\sqrt{M\varpi^3}}{\sqrt{2}(1-\varpi^2)\left(\sinh\sqrt{\frac{\varpi}{2M}}\cos\sqrt{\frac{\varpi}{2M}}+i\cosh\sqrt{\frac{\varpi}{2M}}\sin\sqrt{\frac{\varpi}{2M}}\right) + A\sqrt{M\varpi^3}(i-1)\left(\cosh\sqrt{\frac{\varpi}{2M}}\cos\sqrt{\frac{\varpi}{2M}}+i\sinh\sqrt{\frac{\varpi}{2M}}\sin\sqrt{\frac{\varpi}{2M}}\right)} \\
&= \frac{(-1+i)A\sqrt{M\varpi^3}}{F_1(\varpi)+iF_2(\varpi)} \\
&= \frac{(-1+i)A\sqrt{M\varpi^3}}{F_1(\varpi)+iF_2(\varpi)}\cdot\frac{F_1(\varpi)-iF_2(\varpi)}{F_1(\varpi)-iF_2(\varpi)} \\
&= \frac{(-1+i)A\sqrt{M\varpi^3}[F_1(\varpi)-iF_2(\varpi)]}{[F_1(\varpi)]^2+[F_2(\varpi)]^2} \\
&= \frac{A\sqrt{M\varpi^3}\{[-F_1(\varpi)+F_2(\varpi)]+i[F_1(\varpi)+F_2(\varpi)]\}}{[F_1(\varpi)]^2+[F_2(\varpi)]^2} \\
&= A\sqrt{M\varpi^3}\frac{-F_1(\varpi)+F_2(\varpi)}{[F_1(\varpi)]^2+[F_2(\varpi)]^2} + iA\sqrt{M\varpi^3}\frac{F_1(\varpi)+F_2(\varpi)}{[F_1(\varpi)]^2+[F_2(\varpi)]^2}
\end{aligned}$$

The functions, $F_1(\varpi)$ and $F_2(\varpi)$, are

$$\begin{aligned}
F_1(\varpi) &= \sqrt{2}(1-\varpi^2)\sinh\sqrt{\frac{\varpi}{2M}}\cos\sqrt{\frac{\varpi}{2M}} - A\sqrt{M\varpi^3}\cosh\sqrt{\frac{\varpi}{2M}}\cos\sqrt{\frac{\varpi}{2M}} - A\sqrt{M\varpi^3}\sinh\sqrt{\frac{\varpi}{2M}}\sin\sqrt{\frac{\varpi}{2M}} \\
F_2(\varpi) &= \sqrt{2}(1-\varpi^2)\cosh\sqrt{\frac{\varpi}{2M}}\sin\sqrt{\frac{\varpi}{2M}} + A\sqrt{M\varpi^3}\cosh\sqrt{\frac{\varpi}{2M}}\cos\sqrt{\frac{\varpi}{2M}} - A\sqrt{M\varpi^3}\sinh\sqrt{\frac{\varpi}{2M}}\sin\sqrt{\frac{\varpi}{2M}}.
\end{aligned}$$

Now that the ratio $\theta_R^\circ/\theta_{aR}^\circ$ is in rectangular form, the amplitude ratio and phase shift are known.

$$\left|\frac{\theta_R^\circ}{\theta_{aR}^\circ}\right| = \frac{|\theta_R^\circ|}{\theta_{aR}^\circ} = \sqrt{\left\{A\sqrt{M\varpi^3}\frac{-F_1(\varpi)+F_2(\varpi)}{[F_1(\varpi)]^2+[F_2(\varpi)]^2}\right\}^2 + \left\{A\sqrt{M\varpi^3}\frac{F_1(\varpi)+F_2(\varpi)}{[F_1(\varpi)]^2+[F_2(\varpi)]^2}\right\}^2}$$

$$\tan\alpha = \frac{A\sqrt{M\varpi^3}\frac{F_1(\varpi)+F_2(\varpi)}{[F_1(\varpi)]^2+[F_2(\varpi)]^2}}{A\sqrt{M\varpi^3}\frac{-F_1(\varpi)+F_2(\varpi)}{[F_1(\varpi)]^2+[F_2(\varpi)]^2}} = \frac{F_1(\varpi)+F_2(\varpi)}{-F_1(\varpi)+F_2(\varpi)}$$

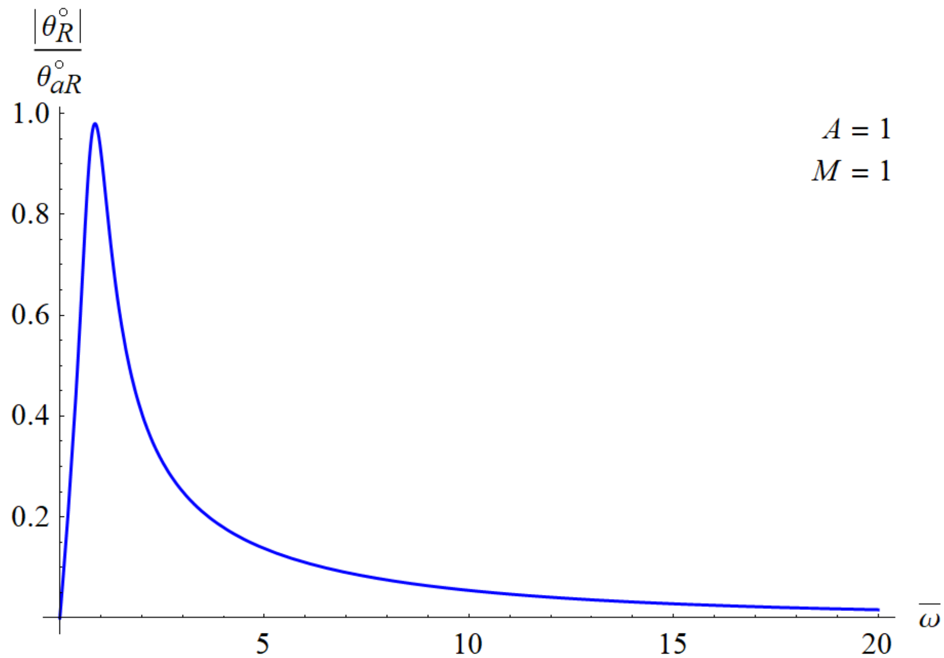
Therefore, plugging in the formulas and fully simplifying,

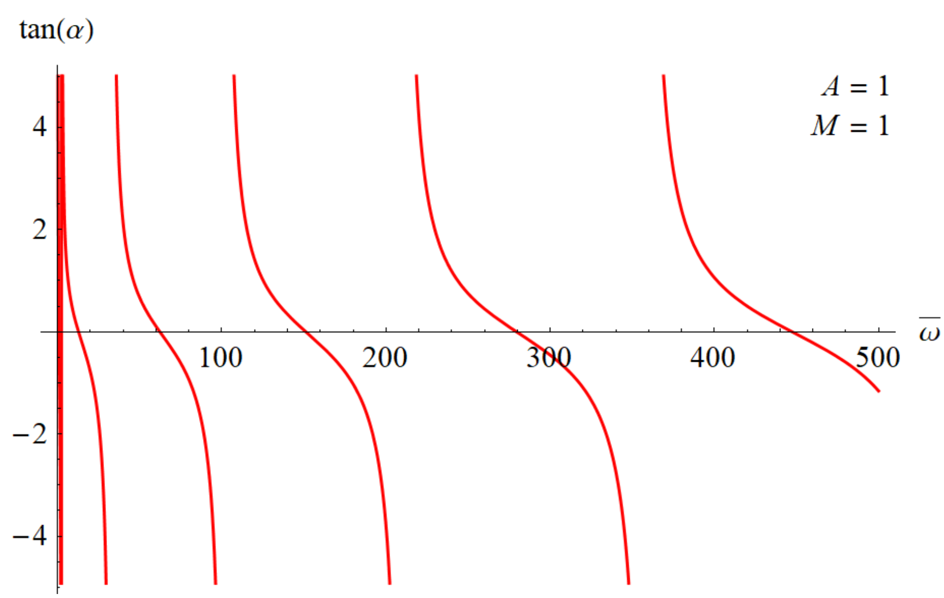
$$\boxed{\frac{|\theta_R^\circ|}{\theta_{aR}^\circ} = \frac{A\sqrt{2M\varpi^3}}{\sqrt{\left[1+\varpi^2(\varpi^2+A^2M\varpi-2)\right]\cosh\sqrt{\frac{2\varpi}{M}} + \sqrt{2}A(\sqrt{M\varpi^7}-\sqrt{M\varpi^3})\left(\sinh\sqrt{\frac{2\varpi}{M}} - \sin\sqrt{\frac{2\varpi}{M}}\right) - \left[1+\varpi^2(\varpi^2-A^2M\varpi-2)\right]\cos\sqrt{\frac{2\varpi}{M}}}}}$$

and

$$\boxed{\tan\alpha = -\frac{\sqrt{2}(\varpi^2-1)\cosh\sqrt{\frac{\varpi}{2M}}\sin\sqrt{\frac{\varpi}{2M}} + \left[\sqrt{2}(\varpi^2-1)\cos\sqrt{\frac{\varpi}{2M}} + 2A\sqrt{M\varpi^3}\sin\sqrt{\frac{\varpi}{2M}}\right]\sinh\sqrt{\frac{\varpi}{2M}}}{2A\sqrt{M\varpi^3}\cos\sqrt{\frac{\varpi}{2M}}\cosh\sqrt{\frac{\varpi}{2M}} + (\varpi^2-1)\left[i^{1/2}\sinh\left(i^{1/2}\sqrt{\frac{\varpi}{M}}\right) + i^{3/2}\sinh\left(i^{3/2}\sqrt{\frac{\varpi}{M}}\right)\right]}.}$$

In order to illustrate the behavior of these functions, the amplitude ratio and phase angle will be plotted versus dimensionless frequency for $A = 1$ and $M = 1$.





The amplitude ratio has a notable maximum at some frequency and then decays to zero, indicating that regardless how fast the cup oscillates, the bob will not follow its motion. The graph of $\tan\alpha$, on the other hand, looks like a graph of cotangent with a period that grows with increasing frequency.

Part (i)

As impressive as these boxed formulas in part (h) are, they're still not good enough because they're unfeasible. They can't be solved for M explicitly for a given amplitude ratio or phase shift, and the maximum amplitude ratio and its corresponding frequency can't be determined easily. What we'll do then is start with Eq. 4C.2-17,

$$\frac{\theta_R^\circ}{\theta_{aR}^\circ} = \frac{AMi\varpi}{(1-\varpi^2)\dfrac{\sinh\sqrt{i\varpi/M}}{\sqrt{i\varpi/M}} + AMi\varpi\cosh\sqrt{i\varpi/M}}, \tag{4C.2-17}$$

and use the power series expansions of hyperbolic sine and hyperbolic cosine with respect to $x = 0$,

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots$$
$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots,$$

in order to get formulas that approximate the boxed ones. Invert both sides of Eq. 4C.2-17.

$$\begin{aligned}\frac{\theta_{aR}^\circ}{\theta_R^\circ} &= \frac{(1-\varpi^2)\dfrac{\sinh\sqrt{i\varpi/M}}{\sqrt{i\varpi/M}} + AMi\varpi\cosh\sqrt{i\varpi/M}}{AMi\varpi} \\ &= \frac{1-\varpi^2}{AMi\varpi}\sqrt{\frac{M}{i\varpi}}\sinh\sqrt{\frac{i\varpi}{M}} + \cosh\sqrt{\frac{i\varpi}{M}}\end{aligned}$$

Provided that M is large, the expansions for cosh and sinh about $x = 0$ can be used here with $\sqrt{i\varpi/M}$ replacing x.

$$\begin{aligned}\frac{\theta_{aR}^\circ}{\theta_R^\circ} &= \frac{1-\varpi^2}{AMi\varpi}\sqrt{\frac{M}{i\varpi}}\left[\left(\sqrt{\frac{i\varpi}{M}}\right) + \frac{1}{6}\left(\sqrt{\frac{i\varpi}{M}}\right)^3 + \frac{1}{120}\left(\sqrt{\frac{i\varpi}{M}}\right)^5 + \frac{1}{5040}\left(\sqrt{\frac{i\varpi}{M}}\right)^7 + \cdots\right] \\ &\qquad + \left[1 + \frac{1}{2}\left(\sqrt{\frac{i\varpi}{M}}\right)^2 + \frac{1}{24}\left(\sqrt{\frac{i\varpi}{M}}\right)^4 + \frac{1}{720}\left(\sqrt{\frac{i\varpi}{M}}\right)^6 + \cdots\right] \\ &= \frac{1-\varpi^2}{AMi\varpi}\left[1 + \frac{1}{6}\left(\sqrt{\frac{i\varpi}{M}}\right)^2 + \frac{1}{120}\left(\sqrt{\frac{i\varpi}{M}}\right)^4 + \frac{1}{5040}\left(\sqrt{\frac{i\varpi}{M}}\right)^6 + \cdots\right] \\ &\qquad + \left[1 + \frac{1}{2}\left(\sqrt{\frac{i\varpi}{M}}\right)^2 + \frac{1}{24}\left(\sqrt{\frac{i\varpi}{M}}\right)^4 + \frac{1}{720}\left(\sqrt{\frac{i\varpi}{M}}\right)^6 + \cdots\right] \\ &= \frac{1-\varpi^2}{AMi\varpi}\left[1 + \frac{1}{6}\left(\frac{i\varpi}{M}\right) + \frac{1}{120}\left(\frac{i\varpi}{M}\right)^2 + \frac{1}{5040}\left(\frac{i\varpi}{M}\right)^3 + \cdots\right] \\ &\qquad + \left[1 + \frac{1}{2}\left(\frac{i\varpi}{M}\right) + \frac{1}{24}\left(\frac{i\varpi}{M}\right)^2 + \frac{1}{720}\left(\frac{i\varpi}{M}\right)^3 + \cdots\right]\end{aligned}$$

Order the terms in powers of $1/M$.

$$\begin{aligned}\frac{\theta_{aR}^{\circ}}{\theta_R^{\circ}} &= 1 + \frac{1}{M}\left[\frac{1-\overline{\omega}^2}{Ai\overline{\omega}} + \frac{1}{2}(i\overline{\omega})\right] + \frac{1}{M^2}\left[\frac{1}{6}\frac{1-\overline{\omega}^2}{Ai\overline{\omega}}(i\overline{\omega}) + \frac{1}{24}(i\overline{\omega})^2\right] + \frac{1}{M^3}\left[\frac{1}{120}\frac{1-\overline{\omega}^2}{Ai\overline{\omega}}(i\overline{\omega})^2 + \frac{1}{720}(i\overline{\omega})^3\right] \\ &\quad + \frac{1}{M^4}\left[\frac{1}{5040}\frac{1-\overline{\omega}^2}{Ai\overline{\omega}}(i\overline{\omega})^3 + \frac{1}{40\,320}(i\overline{\omega})^4\right] + \frac{1}{M^5}\left[\frac{1}{362\,880}\frac{1-\overline{\omega}^2}{Ai\overline{\omega}}(i\overline{\omega})^4 + \frac{1}{3\,628\,800}(i\overline{\omega})^5\right] + \cdots \\ &= 1 + \frac{i}{M}\left(\frac{\overline{\omega}^2-1}{A\overline{\omega}} + \frac{\overline{\omega}}{2}\right) - \frac{1}{M^2}\left(\frac{\overline{\omega}^2-1}{6A} + \frac{\overline{\omega}^2}{24}\right) - \frac{i}{M^3}\left(\frac{\overline{\omega}}{120}\frac{\overline{\omega}^2-1}{A} + \frac{\overline{\omega}^3}{720}\right) \\ &\quad + \frac{1}{M^4}\left(\frac{\overline{\omega}^2}{5040}\frac{\overline{\omega}^2-1}{A} + \frac{\overline{\omega}^4}{40\,320}\right) + \frac{i}{M^5}\left(\frac{\overline{\omega}^3}{362\,880}\frac{\overline{\omega}^2-1}{A} + \frac{\overline{\omega}^5}{3\,628\,800}\right) - \cdots \\ &= F_3(\overline{\omega}) + iF_4(\overline{\omega})\end{aligned}$$

The amplitude ratio is obtained by taking the absolute value of both sides

$$\left|\frac{\theta_{aR}^{\circ}}{\theta_R^{\circ}}\right| = |F_3(\overline{\omega}) + iF_4(\overline{\omega})|$$

$$\frac{\theta_{aR}^{\circ}}{|\theta_R^{\circ}|} = \sqrt{[F_3(\overline{\omega})]^2 + [F_4(\overline{\omega})]^2}$$

and then inverting both sides to get the output-to-input ratio specifically.

$$\boxed{\frac{|\theta_R^{\circ}|}{\theta_{aR}^{\circ}} = \frac{1}{\sqrt{[F_3(\overline{\omega})]^2 + [F_4(\overline{\omega})]^2}}}$$

On the other hand, the phase shift is obtained by inverting both sides first

$$\left(\frac{\theta_{aR}^{\circ}}{\theta_R^{\circ}}\right)^{-1} = [F_3(\overline{\omega}) + iF_4(\overline{\omega})]^{-1}$$

$$\frac{\theta_R^{\circ}}{\theta_{aR}^{\circ}} = \frac{F_3(\overline{\omega})}{[F_3(\overline{\omega})]^2 + [F_4(\overline{\omega})]^2} - i\frac{F_4(\overline{\omega})}{[F_3(\overline{\omega})]^2 + [F_4(\overline{\omega})]^2}$$

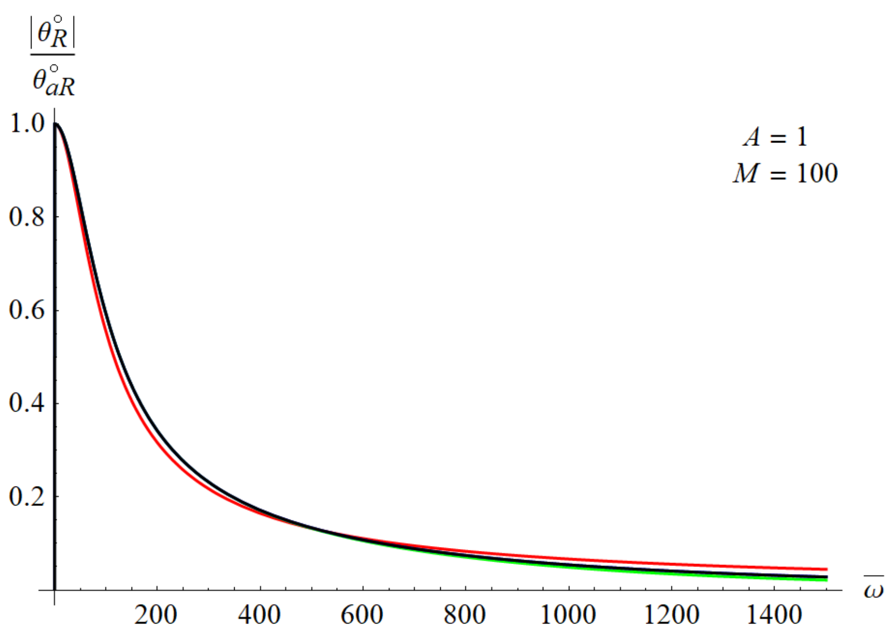
and then writing

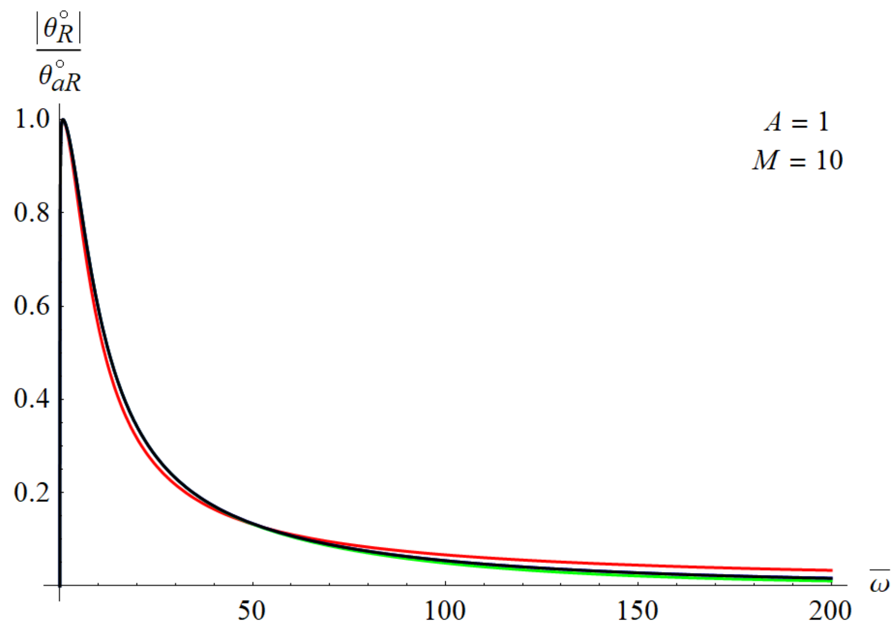
$$\tan\alpha = \frac{-\frac{F_4(\overline{\omega})}{[F_3(\overline{\omega})]^2 + [F_4(\overline{\omega})]^2}}{\frac{F_3(\overline{\omega})}{[F_3(\overline{\omega})]^2 + [F_4(\overline{\omega})]^2}}$$

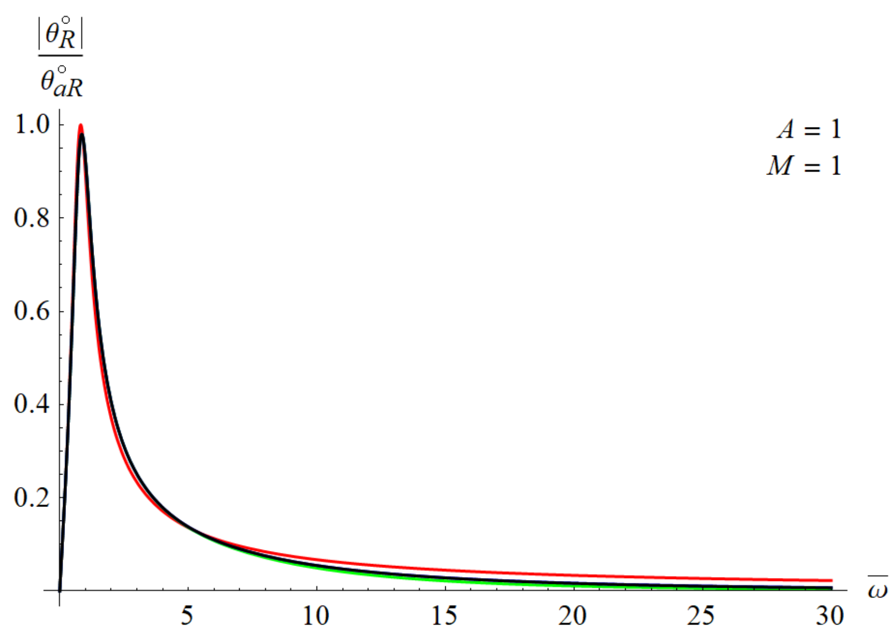
$$\boxed{\tan\alpha = -\frac{F_4(\overline{\omega})}{F_3(\overline{\omega})}.}$$

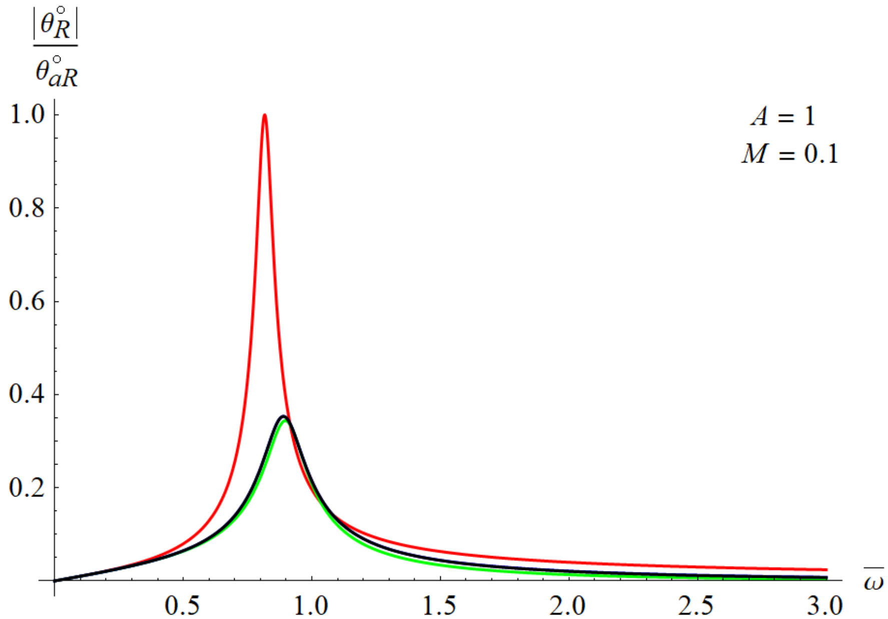
The more terms that are included in $F_3(\overline{\omega})$ and $F_4(\overline{\omega})$, the better these boxed formulas will approximate the amplitude ratio and phase shift. These next graphs show side-by-side comparisons of these boxed formulas with those in part (h) for $A = 1$ and various values of M to see how large M has to be for these approximations to be useful. In black is the graph of the boxed formula in part (h) and in red, green, and blue are graphs of the boxed formulas here with two, four, and six terms from the power series expansion, respectively.

$$F_3(\varpi) = 1$$
$$F_4(\varpi) = \frac{1}{M}\left(\frac{\varpi^2 - 1}{A\varpi} + \frac{\varpi}{2}\right)$$
$$F_3(\varpi) = 1 - \frac{1}{M^2}\left(\frac{\varpi^2 - 1}{6A} + \frac{\varpi^2}{24}\right)$$
$$F_4(\varpi) = \frac{1}{M}\left(\frac{\varpi^2 - 1}{A\varpi} + \frac{\varpi}{2}\right) - \frac{1}{M^3}\left(\frac{\varpi}{120}\frac{\varpi^2 - 1}{A} + \frac{\varpi^3}{720}\right)$$
$$F_3(\varpi) = 1 - \frac{1}{M^2}\left(\frac{\varpi^2 - 1}{6A} + \frac{\varpi^2}{24}\right) + \frac{1}{M^4}\left(\frac{\varpi^2}{5040}\frac{\varpi^2 - 1}{A} + \frac{\varpi^4}{40\,320}\right)$$
$$F_4(\varpi) = \frac{1}{M}\left(\frac{\varpi^2 - 1}{A\varpi} + \frac{\varpi}{2}\right) - \frac{1}{M^3}\left(\frac{\varpi}{120}\frac{\varpi^2 - 1}{A} + \frac{\varpi^3}{720}\right) + \frac{1}{M^5}\left(\frac{\varpi^3}{362\,880}\frac{\varpi^2 - 1}{A} + \frac{\varpi^5}{3\,628\,880}\right)$$

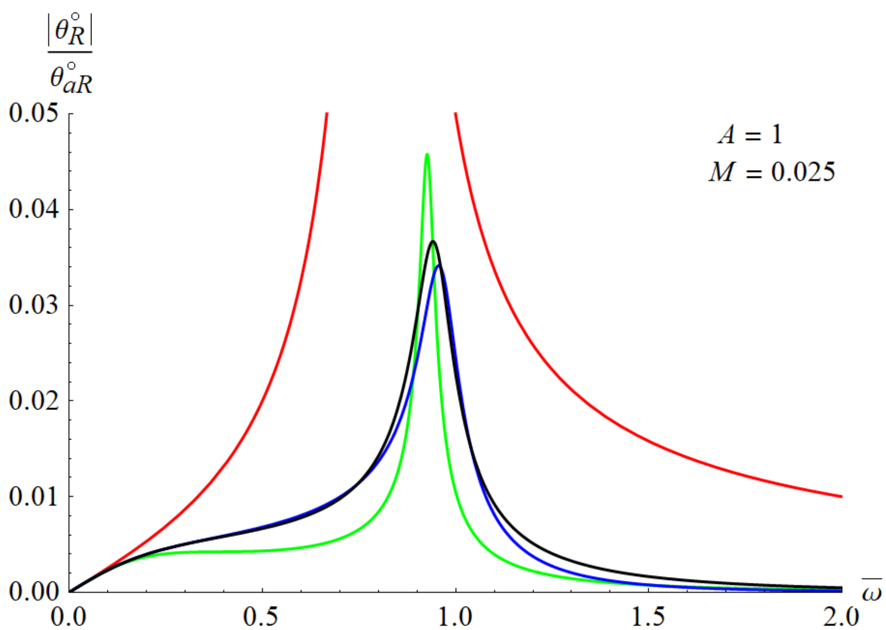



$\frac{|\theta_R^\circ|}{\theta_{aR}^\circ}$
1.0
0.8
0.6
0.4
0.2
$A = 1$
$M = 10$
50
100
150
200
$\overline{\omega}$

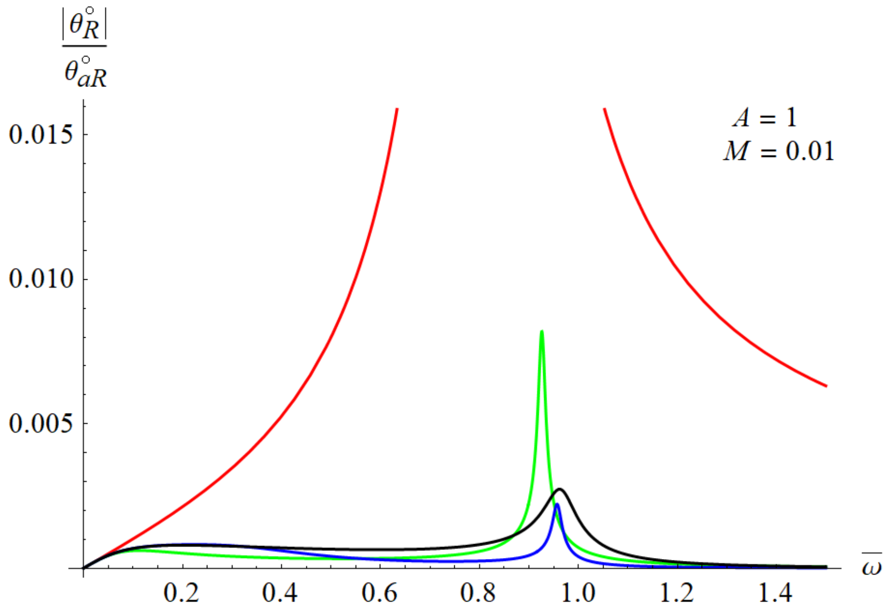

$\frac{|\theta_R^\circ|}{\theta_{aR}^\circ}$
1.0
0.8
0.6
0.4
0.2
$A = 1$
$M = 1$
5
10
15
20
25
30
$\overline{\omega}$

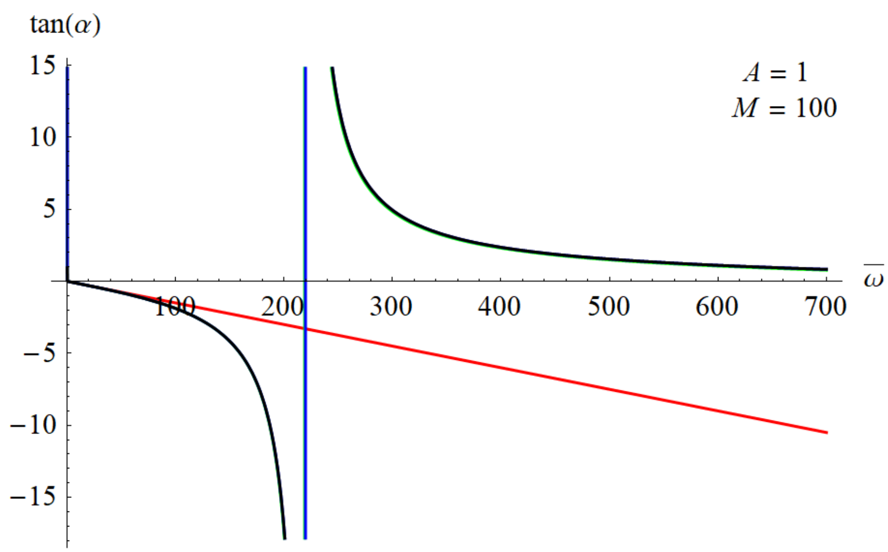


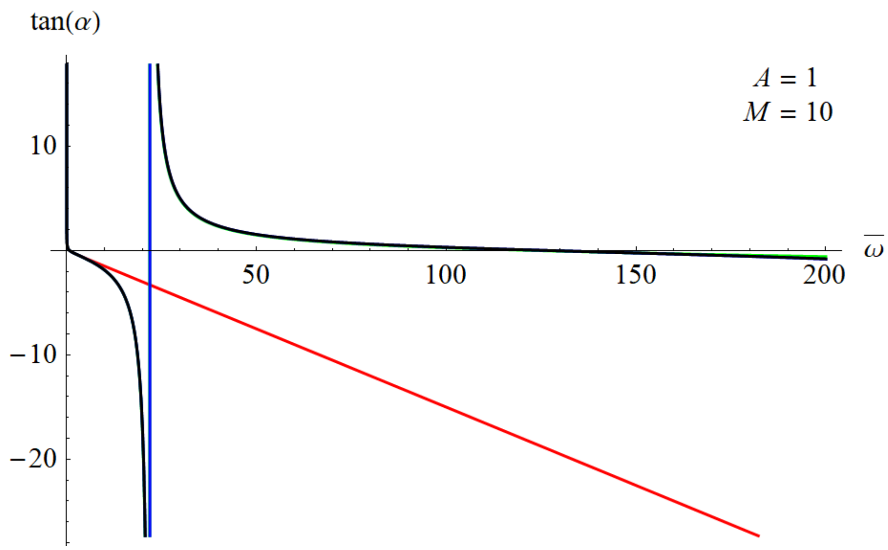
$\frac{|\theta_R^\circ|}{\theta_{aR}^\circ}$
1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0
2.5
3.0
$\overline{\omega}$
$A = 1$
$M = 0.1$

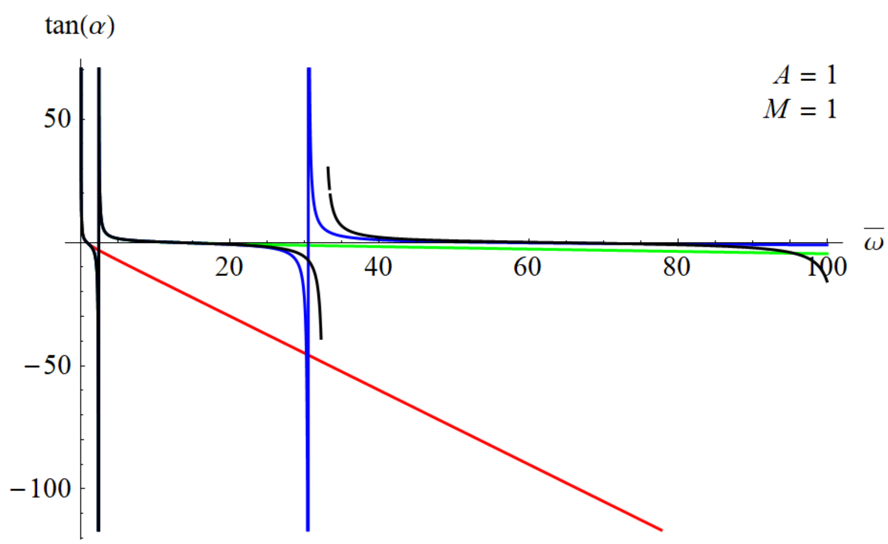


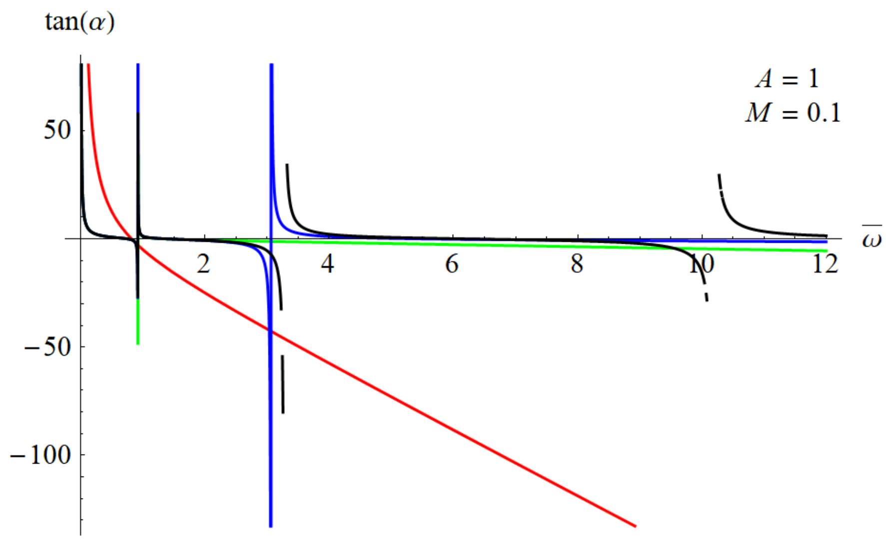
$\frac{|\theta_R^\circ|}{\theta_{aR}^\circ}$
0.05
0.04
0.03
0.02
0.01
0.00
0.0
0.5
1.0
1.5
2.0
$\overline{\omega}$
$A = 1$
$M = 0.025$

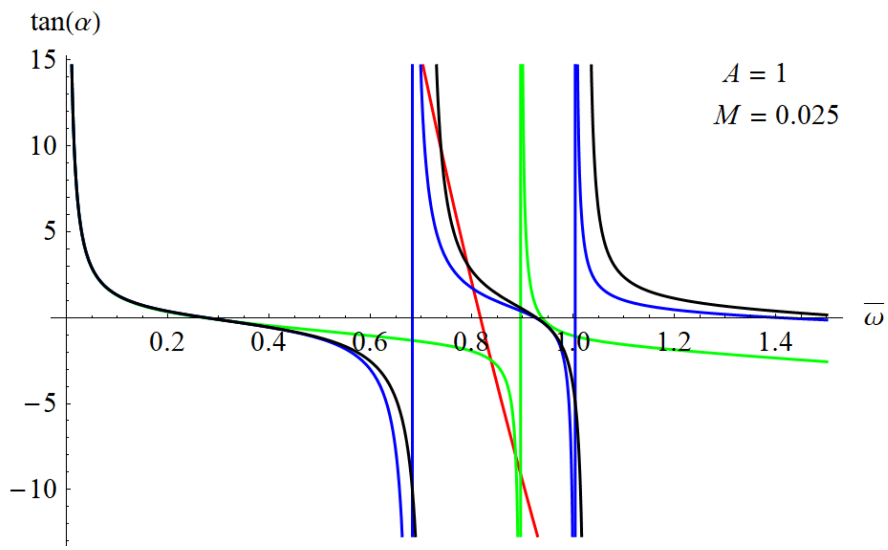

$\frac{|\theta_R^\circ|}{\theta_{aR}^\circ}$
0.015
0.010
0.005
0.2
0.4
0.6
0.8
1.0
1.2
1.4
$\bar{\omega}$
$A = 1$
$M = 0.01$

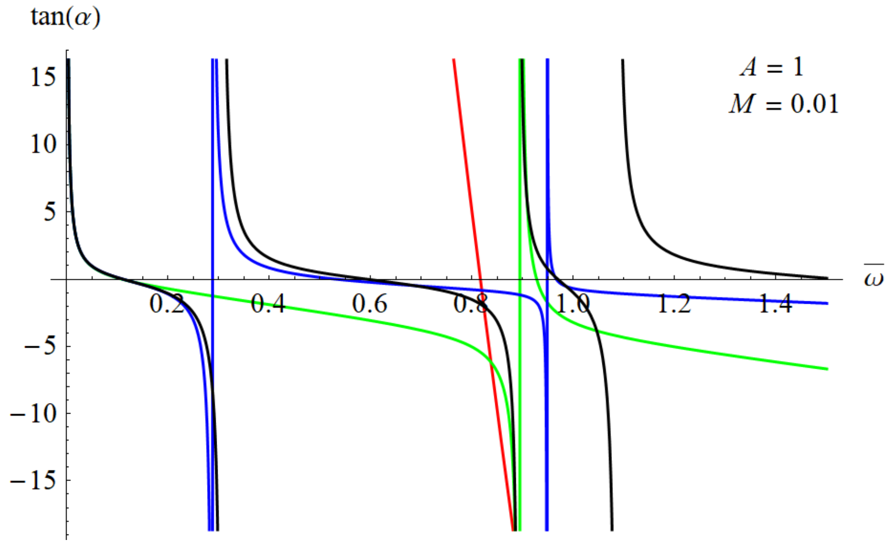

tan(α)
15
10
5
−5
−10
−15
100
200
300
400
500
600
700
$\bar{\omega}$
$A = 1$
$M = 100$


tan(α)
A = 1
M = 10
10
−10
−20
50
100
150
200
ω̄


tan(α)
A = 1
M = 1
50
−50
−100
20
40
60
80
100
ω̄


tan(α)
50
−50
−100
2
4
6
8
10
12
ω̄
A = 1
M = 0.1


tan(α)
15
10
5
−5
−10
0.2
0.4
0.6
0.8
1.0
1.2
1.4
ω̄
A = 1
M = 0.025



The point of these graphs is to show that the smaller M is, the more terms in $F_3(\overline{\omega})$ and $F_4(\overline{\omega})$ one needs for the approximations to have a reasonable level of accuracy. Assuming the fluid has high enough viscosity ($M \geq 1$, judging from the graphs), the amplitude ratio can be approximated by the first two terms in the power series.

$$\begin{aligned}\frac{|\theta_R^\circ|}{\theta_{aR}^\circ} &= \frac{1}{\sqrt{[F_3(\overline{\omega})]^2 + [F_4(\overline{\omega})]^2}} \\ &= \frac{1}{\sqrt{[(1)]^2 + \left[\frac{1}{M}\left(\frac{\overline{\omega}^2-1}{A\overline{\omega}} + \frac{\overline{\omega}}{2}\right)\right]^2}}\end{aligned}$$

Solving for M, the dimensionless viscosity is

$$M = \frac{\frac{\overline{\omega}^2-1}{A\overline{\omega}} + \frac{\overline{\omega}}{2}}{\sqrt{\left(\frac{\theta_{aR}^\circ}{|\theta_R^\circ|}\right)^2 - 1}}.$$

Therefore, plugging in the formulas for M, A, and $\overline{\omega}$ and simplifying, the fluid viscosity is

$$\boxed{\mu = (a-1)\frac{\omega^2\left[\pi R^4 \rho L(a-1) + I\right] - k}{2\pi R^2 \omega L\sqrt{\left(\frac{\theta_{aR}^\circ}{|\theta_R^\circ|}\right)^2 - 1}}}$$

if $\mu \geq \rho(a-1)^2 R^2\sqrt{k/I}$. To determine the maximum of the amplitude ratio, take the derivative with respect to $\overline{\omega}$

$$\frac{d(|\theta_R^\circ|/\theta_{aR}^\circ)}{d\overline{\omega}} = \frac{\left(\frac{\overline{\omega}^2-1}{A\overline{\omega}^2} - \frac{2}{A} - \frac{1}{2}\right)\left(\frac{\overline{\omega}^2-1}{A\overline{\omega}} + \frac{\overline{\omega}}{2}\right)}{M^2\left[1 + \frac{1}{M^2}\left(\frac{\overline{\omega}^2-1}{A\overline{\omega}} + \frac{\overline{\omega}}{2}\right)^2\right]^{3/2}}$$

and then set it equal to zero.

$$\left(\frac{\overline{\omega}^2-1}{A\overline{\omega}^2} - \frac{2}{A} - \frac{1}{2}\right)\left(\frac{\overline{\omega}^2-1}{A\overline{\omega}} + \frac{\overline{\omega}}{2}\right) = 0$$

Solve for $\overline{\omega}$.

$$\overline{\omega} = \left\{\pm\sqrt{\frac{2}{A+2}}, \pm i\sqrt{\frac{2}{A+2}}\right\}$$

The dimensionless frequency at which the amplitude ratio is maximum is

$$\overline{\omega}_{\text{max}} = \sqrt{\frac{2}{A+2}} \quad \Rightarrow \quad \frac{|\theta_R^\circ|}{\theta_{aR}^\circ}(\overline{\omega}_{\text{max}}) = 1.$$

Therefore, plugging in the formulas for A and $\overline{\omega}$ and simplifying, the frequency that results in the maximum amplitude ratio is

$$\boxed{\omega_{\text{max}} = \sqrt{\frac{k}{\pi R^4 \rho L(a-1) + I}}}$$

if $\mu \geq \rho(a-1)^2 R^2 \sqrt{k/I}$.

Part (j)

Note that a dyne is the force needed to give 1 gram an acceleration of 1 cm/s^2. Use this fact to rewrite k.

$$k = 4 \times 10^6 \text{ dyn cm} = 4 \times 10^6 \ \frac{\text{g} \cdot \text{cm}^2}{\text{s}^2}$$

Now that all the given quantities are in grams, centimeters, and seconds, A and M will be dimensionless as desired.

$$A = \frac{2\pi R^4 L\rho(a-1)}{I} = \frac{2\pi(5.5 \text{ cm})^4(25 \text{ cm})\left(1 \ \frac{\text{g}}{\text{cm}^3}\right)\left(\frac{10^{-2} \text{ cm}}{5.5 \text{ cm}}\right)}{2500 \text{ g} \cdot \text{cm}^2} \approx 0.1045$$

$$M = \frac{\mu/\rho}{(a-1)^2 R^2}\sqrt{\frac{I}{k}} = \frac{10^{-2} \text{ cm}^2/\text{s}}{(10^{-2} \text{ cm})^2}\sqrt{\frac{2500 \text{ g} \cdot \text{cm}^2}{4 \times 10^6 \ \frac{\text{g}\cdot\text{cm}^2}{\text{s}^2}}} = \frac{5}{2} = 2.5$$

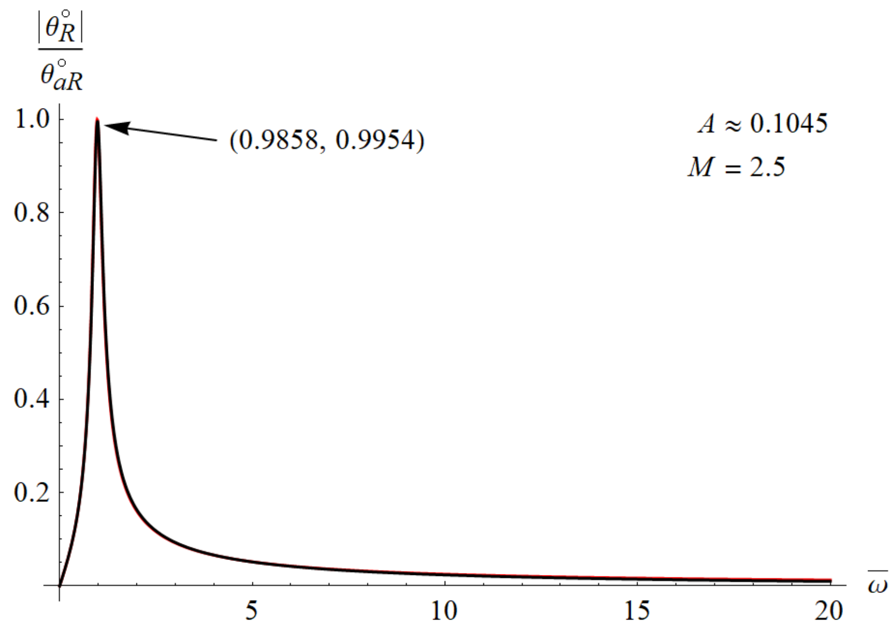
Since $M \geq 1$, the formulas derived in part (i) are valid. The amplitude ratio is predicted to be maximum at

$$\overline{\omega}_{\text{max}} = \sqrt{\frac{2}{A+2}} \approx 0.9748$$

and have a value of 1 there. On the following graph of amplitude ratio versus dimensionless frequency, the black curve represents the boxed formula in part (h), and the red curve represents the boxed formula in part (i) with two terms from the power series expansion.

$$F_3(\overline{\omega}) = 1$$

$$F_4(\overline{\omega}) = \frac{1}{M}\left(\frac{\overline{\omega}^2 - 1}{A\overline{\omega}} + \frac{\overline{\omega}}{2}\right)$$



The point labelled is the maximum of the black curve.