Problem 4C.4

Radial flow through a porous medium (Fig. 4C.4). A fluid flows through a porous cylindrical shell with inner and outer radii R_1 and R_2, respectively. At these surfaces, the pressures are known to be p_1 and p_2, respectively. The length of the cylindrical shell is h.

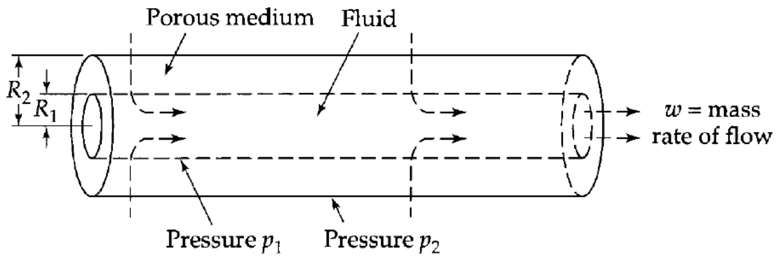


Fig. 4C.4. Radial flow through a porous medium.

(a) Find the pressure distribution, radial flow velocity, and mass rate of flow for an incompressible fluid.

(b) Rework (a) for a compressible liquid and for an ideal gas.

Answers: **(a)** $\dfrac{\mathscr{P}-\mathscr{P}_1}{\mathscr{P}_2-\mathscr{P}_1}=\dfrac{\ln(r/R_1)}{\ln(R_2/R_1)}$ $\quad v_{0r}=-\dfrac{\kappa}{\mu r}\dfrac{\mathscr{P}_2-\mathscr{P}_1}{\ln(R_2/R_1)}$ $\quad w=\dfrac{2\pi\kappa h(\mathscr{P}_2-\mathscr{P}_1)\rho}{\mu\ln(R_2/R_1)}$

Solution

Case 1: An Incompressible Fluid

According to Problem 4C.3, the modified pressure $\mathscr{P}$ (defined so as to satisfy $\nabla\mathscr{P}=\nabla p-\rho\mathbf{g}$) satisfies the Laplace equation,

$$\nabla^2\mathscr{P}=0,$$

in an incompressible fluid for porous flow. For radial flow specifically, the modified pressure is only a function of r: $\mathscr{P}=\mathscr{P}(r)$. Looking at Fig. 4C.4, it is subject to the following boundary conditions.

$$\mathscr{P}(R_1)=\mathscr{P}_1$$
$$\mathscr{P}(R_2)=\mathscr{P}_2$$

Using formula (B) on page 834, expand the Laplacian operator in cylindrical coordinates.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\mathscr{P}}{\partial r}\right)+\underbrace{\frac{1}{r^2}\frac{\partial^2\mathscr{P}}{\partial\theta^2}}_{=0}+\underbrace{\frac{\partial^2\mathscr{P}}{\partial z^2}}_{=0}=0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathscr{P}}{dr}\right)=0$$

Multiply both sides by r.

$$\frac{d}{dr}\left(r\frac{d\mathscr{P}}{dr}\right)=0$$

Integrate both sides with respect to r.

$$r\frac{d\mathscr{P}}{dr} = C_1$$

Divide both sides by r.

$$\frac{d\mathscr{P}}{dr} = \frac{C_1}{r}$$

Integrate both sides with respect to r once more.

$$\mathscr{P}(r) = C_1 \ln r + C_2$$

Apply the boundary conditions here to determine C_1 and C_2.

$$\mathscr{P}(R_1) = C_1 \ln R_1 + C_2 = \mathscr{P}_1$$
$$\mathscr{P}(R_2) = C_1 \ln R_2 + C_2 = \mathscr{P}_2$$

Solving this system of equations yields

$$C_1 = \frac{\mathscr{P}_2 - \mathscr{P}_1}{\ln(R_2/R_1)} \quad \text{and} \quad C_2 = \mathscr{P}_1 - \frac{\mathscr{P}_2 - \mathscr{P}_1}{\ln(R_2/R_1)} \ln R_1.$$

So then

$$\mathscr{P}(r) = \frac{\mathscr{P}_2 - \mathscr{P}_1}{\ln(R_2/R_1)} \ln r + \mathscr{P}_1 - \frac{\mathscr{P}_2 - \mathscr{P}_1}{\ln(R_2/R_1)} \ln R_1.$$

Subtract $\mathscr{P}_1$ from both sides.

$$\mathscr{P}(r) - \mathscr{P}_1 = \frac{\mathscr{P}_2 - \mathscr{P}_1}{\ln(R_2/R_1)} \ln r - \frac{\mathscr{P}_2 - \mathscr{P}_1}{\ln(R_2/R_1)} \ln R_1$$

Divide both sides by $\mathscr{P}_2 - \mathscr{P}_1$.

$$\begin{aligned}\frac{\mathscr{P}(r) - \mathscr{P}_1}{\mathscr{P}_2 - \mathscr{P}_1} &= \frac{1}{\ln(R_2/R_1)} \ln r - \frac{1}{\ln(R_2/R_1)} \ln R_1 \\ &= \frac{\ln r - \ln R_1}{\ln(R_2/R_1)}\end{aligned}$$

Therefore,

$$\boxed{\frac{\mathscr{P} - \mathscr{P}_1}{\mathscr{P}_2 - \mathscr{P}_1} = \frac{\ln(r/R_1)}{\ln(R_2/R_1)}.}$$

The radial flow velocity is obtained from Darcy's equation.

$$\begin{aligned}\mathbf{v}_0 &= -\frac{\kappa}{\mu}(\nabla p - \rho\mathbf{g}) \\ &= -\frac{\kappa}{\mu}(\nabla \mathscr{P}) \\ &= -\frac{\kappa}{\mu}\frac{d\mathscr{P}}{dr}\hat{\mathbf{r}} \\ &= -\frac{\kappa}{\mu}\frac{C_1}{r}\hat{\mathbf{r}} \\ &= -\frac{\kappa}{\mu r}\frac{\mathscr{P}_2 - \mathscr{P}_1}{\ln(R_2/R_1)}\hat{\mathbf{r}}\end{aligned}$$

Therefore,

$$\boxed{v_{0r} = -\frac{\kappa}{\mu r}\frac{\mathscr{P}_2 - \mathscr{P}_1}{\ln(R_2/R_1)}.}$$

Calculate the volumetric flow rate through a cylindrical shell of radius r, where $R_1 \leq r \leq R_2$.

$$\begin{aligned}\frac{dV}{dt} &= \mathbf{v} \cdot \mathbf{A} \\ &= (v_{0r}\hat{\mathbf{r}}) \cdot (2\pi r h \hat{\mathbf{r}}) \\ &= 2\pi h r v_{0r} \\ &= -\frac{2\pi\kappa h}{\mu} \frac{\mathscr{P}_2 - \mathscr{P}_1}{\ln(R_2/R_1)}\end{aligned}$$

Multiply both sides by the fluid density ρ to get the mass flow rate through this shell.

$$\rho \frac{dV}{dt} = -\frac{2\pi\kappa h}{\mu} \frac{\mathscr{P}_2 - \mathscr{P}_1}{\ln(R_2/R_1)} \rho$$

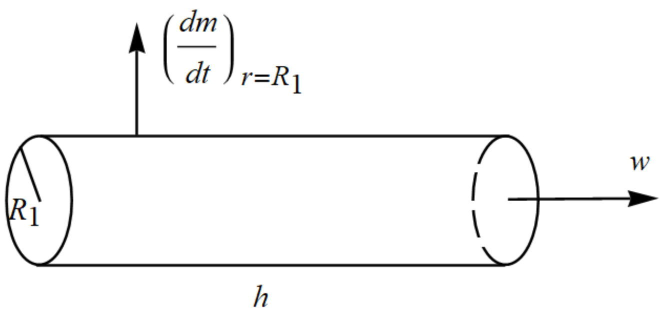
ρ is constant, so it can be brought inside the derivative.

$$\frac{d(\rho V)}{dt} = -\frac{2\pi\kappa h}{\mu} \frac{\mathscr{P}_2 - \mathscr{P}_1}{\ln(R_2/R_1)} \rho$$

Density times volume is mass.

$$\frac{dm}{dt} = -\frac{2\pi\kappa h}{\mu} \frac{\mathscr{P}_2 - \mathscr{P}_1}{\ln(R_2/R_1)} \rho$$

Here dm/dt represents the mass flow rate out of the shell due to the radial velocity $\mathbf{v} = v_{0r}(r)\hat{\mathbf{r}}$. Now consider the cylindrical shell of radius $r = R_1$ in particular.



Construct a mass balance over it.

$$\text{rate of mass in} - \text{rate of mass out} = \text{rate of mass accumulation}$$

Everything that flows in must flow out, so the accumulation rate is zero.

$$\text{rate of mass in} - \text{rate of mass out} = 0$$

$$0 - \left(\left.\frac{dm}{dt}\right|_{r=R_1} + w \right) = 0$$

$$w = -\left.\frac{dm}{dt}\right|_{r=R_1}$$

Therefore,

$$\boxed{w = \frac{2\pi\kappa h(\mathscr{P}_2 - \mathscr{P}_1)\rho}{\mu \ln(R_2/R_1)}.}$$

Case 2: A Compressible Liquid

According to Problem 4C.3, the fluid density ρ satisfies

$$\left(\frac{\varepsilon\mu\beta}{\kappa}\right)\frac{\partial\rho}{\partial t} = \nabla^2\rho - (\nabla\cdot\rho^2\beta\mathbf{g})$$

in a compressible liquid for porous flow. Substitute the equation of state $\rho = \rho_0 e^{\beta p}$ to get an equation for the pressure.

$$\frac{\varepsilon\mu\beta}{\kappa}\frac{\partial}{\partial t}(\rho_0 e^{\beta p}) = \nabla^2(\rho_0 e^{\beta p}) - \nabla\cdot(\rho_0 e^{\beta p})^2\beta\mathbf{g}$$

$$\frac{\varepsilon\mu\beta}{\kappa}\rho_0 e^{\beta p}\beta\frac{\partial p}{\partial t} = \rho_0\nabla^2(e^{\beta p}) - \rho_0^2\beta\nabla\cdot e^{2\beta p}\mathbf{g}$$

Expand the divergence and Laplacian in cylindrical coordinates by using formulas (A) and (B), respectively, on page 834.

$$\begin{aligned}\frac{\varepsilon\mu\beta}{\kappa}\rho_0 e^{\beta p}\beta\frac{\partial p}{\partial t} = \rho_0\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}e^{\beta p}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}e^{\beta p} + \frac{\partial^2}{\partial z^2}e^{\beta p}\right] \\ - \rho_0^2\beta\left[\frac{1}{r}\frac{\partial}{\partial r}(re^{2\beta p}g_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(e^{2\beta p}g_\theta) + \frac{\partial}{\partial z}(e^{2\beta p}g_z)\right]\end{aligned}$$

For the radial flow in this problem, the pressure is only a function of r: $p = p(r)$. All derivatives except those with respect to r vanish as a result.

$$0 = \rho_0\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}e^{\beta p}\right)\right] - \rho_0^2\beta\left[\frac{1}{r}\frac{d}{dr}(re^{2\beta p}g_r)\right]$$

Gravity has no component in the radial direction, so $g_r = 0$.

$$\rho_0\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}e^{\beta p}\right)\right] = 0$$

Divide both sides by ρ_0.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}e^{\beta p}\right) = 0$$

Multiply both sides by r.

$$\frac{d}{dr}\left(r\frac{d}{dr}e^{\beta p}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{d}{dr}e^{\beta p} = C_3$$

Divide both sides by r.

$$\frac{d}{dr}e^{\beta p} = \frac{C_3}{r}$$

Integrate both sides with respect to r once more.

$$e^{\beta p} = C_3\ln r + C_4$$

Apply the two boundary conditions,

$$p(R_1) = p_1$$
$$p(R_2) = p_2,$$

in order to determine C_3 and C_4.

$$e^{\beta p_1} = C_3 \ln R_1 + C_4$$
$$e^{\beta p_2} = C_3 \ln R_2 + C_4$$

Solving this system of equations yields

$$C_3 = \frac{e^{\beta p_2} - e^{\beta p_1}}{\ln(R_2/R_1)} \quad \text{and} \quad C_4 = \frac{e^{\beta p_1}\ln R_2 - e^{\beta p_2}\ln R_1}{\ln(R_2/R_1)}.$$

So then

$$\begin{aligned} e^{\beta p} &= \frac{e^{\beta p_2} - e^{\beta p_1}}{\ln(R_2/R_1)}\ln r + \frac{e^{\beta p_1}\ln R_2 - e^{\beta p_2}\ln R_1}{\ln(R_2/R_1)} \\ &= \frac{e^{\beta p_2}\ln r - e^{\beta p_1}\ln r + e^{\beta p_1}\ln R_2 - e^{\beta p_2}\ln R_1}{\ln(R_2/R_1)} \\ &= \frac{e^{\beta p_2}(\ln r - \ln R_1) + e^{\beta p_1}(\ln R_2 - \ln r)}{\ln(R_2/R_1)} \\ &= \frac{e^{\beta p_2}\ln(r/R_1) + e^{\beta p_1}\ln(R_2/r)}{\ln(R_2/R_1)}. \end{aligned}$$

Take the natural logarithm of both sides.

$$\beta p = \ln \frac{e^{\beta p_2}\ln(r/R_1) + e^{\beta p_1}\ln(R_2/r)}{\ln(R_2/R_1)}$$

Divide both sides by β.

$$p(r) = \frac{1}{\beta}\ln \frac{e^{\beta p_2}\ln(r/R_1) + e^{\beta p_1}\ln(R_2/r)}{\ln(R_2/R_1)}$$

Therefore,

$$\boxed{p(r) = \ln\left[\frac{e^{\beta p_2}\ln(r/R_1) + e^{\beta p_1}\ln(R_2/r)}{\ln(R_2/R_1)}\right]^{1/\beta}.}$$

The radial flow velocity is obtained from Darcy's equation.

$$\begin{aligned} \mathbf{v}_0 &= -\frac{\kappa}{\mu}(\nabla p - \rho\mathbf{g}) \\ &= -\frac{\kappa}{\mu}\left(\frac{dp}{dr}\hat{\mathbf{r}} - \rho g_z \hat{\mathbf{z}}\right) \\ &= -\frac{\kappa}{\mu}\left\{\frac{1}{\beta}\left[\frac{e^{\beta p_2}\ln(r/R_1) + e^{\beta p_1}\ln(R_2/r)}{\ln(R_2/R_1)}\right]^{-1}\left[\frac{e^{\beta p_2}(1/r) + e^{\beta p_1}(-1/r)}{\ln(R_2/R_1)}\right]\hat{\mathbf{r}} - \rho g_z\hat{\mathbf{z}}\right\} \\ &= -\frac{\kappa}{\mu\beta r}\frac{e^{\beta p_2} - e^{\beta p_1}}{e^{\beta p_2}\ln(r/R_1) + e^{\beta p_1}\ln(R_2/r)}\hat{\mathbf{r}} + \frac{\kappa\rho}{\mu}g_z\hat{\mathbf{z}} \end{aligned}$$

Therefore,

$$\boxed{v_{0r} = -\frac{\kappa}{\mu\beta r}\frac{e^{\beta p_2} - e^{\beta p_1}}{e^{\beta p_2}\ln(r/R_1) + e^{\beta p_1}\ln(R_2/r)}.}$$

Calculate the volumetric flow rate through a cylindrical shell of radius r, where $R_1 \le r \le R_2$.

$$\begin{aligned}\frac{dV}{dt} &= \mathbf{v}\cdot\mathbf{A} \\ &= (v_{0r}\hat{\mathbf{r}})\cdot(2\pi r h\hat{\mathbf{r}}) \\ &= 2\pi h r v_{0r} \\ &= -\frac{2\pi\kappa h}{\mu\beta}\frac{e^{\beta p_2} - e^{\beta p_1}}{e^{\beta p_2}\ln(r/R_1) + e^{\beta p_1}\ln(R_2/r)}\end{aligned}$$

Multiply both sides by the fluid density ρ to get the mass flow rate through this shell.

$$\rho\frac{dV}{dt} = -\frac{2\pi\kappa h}{\mu\beta}\frac{e^{\beta p_2} - e^{\beta p_1}}{e^{\beta p_2}\ln(r/R_1) + e^{\beta p_1}\ln(R_2/r)}\rho$$

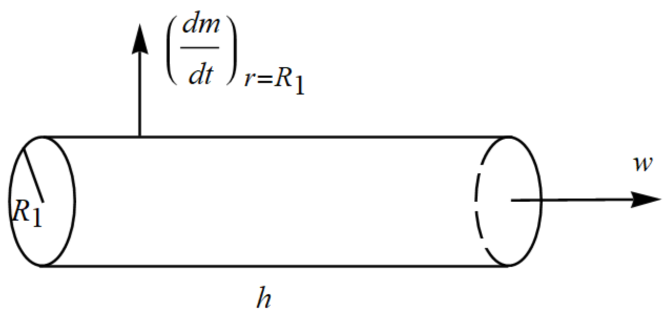
ρ is constant, so it can be brought inside the derivative.

$$\frac{d(\rho V)}{dt} = -\frac{2\pi\kappa h}{\mu\beta}\frac{e^{\beta p_2} - e^{\beta p_1}}{e^{\beta p_2}\ln(r/R_1) + e^{\beta p_1}\ln(R_2/r)}\rho$$

Density times volume is mass.

$$\frac{dm}{dt} = -\frac{2\pi\kappa h}{\mu\beta}\frac{e^{\beta p_2} - e^{\beta p_1}}{e^{\beta p_2}\ln(r/R_1) + e^{\beta p_1}\ln(R_2/r)}\rho$$

Here dm/dt represents the mass flow rate out of the shell due to the radial velocity $\mathbf{v} = v_{0r}(r)\hat{\mathbf{r}}$. Now consider the cylindrical shell of radius $r = R_1$ in particular.



Construct a mass balance over it.

$$\text{rate of mass in} - \text{rate of mass out} = \text{rate of mass accumulation}$$

Everything that flows in must flow out, so the accumulation rate is zero.

$$\text{rate of mass in} - \text{rate of mass out} = 0$$

$$0 - \left(\left.\frac{dm}{dt}\right|_{r=R_1} + w\right) = 0$$

Solve for w.

$$\begin{aligned} w &= -\frac{dm}{dt}\bigg|_{r=R_1} \\ &= \frac{2\pi\kappa h}{\mu\beta}\frac{e^{\beta p_2} - e^{\beta p_1}}{e^{\beta p_2}\ln(R_1/R_1) + e^{\beta p_1}\ln(R_2/R_1)}\rho \\ &= \frac{2\pi\kappa h}{\mu\beta}\frac{e^{\beta p_2} - e^{\beta p_1}}{e^{\beta p_1}\ln(R_2/R_1)}\rho \\ &= \frac{2\pi\kappa h}{\mu\beta}\frac{e^{\beta p_2 - \beta p_1} - 1}{\ln(R_2/R_1)}\rho \end{aligned}$$

Therefore,

$$\boxed{w = \frac{2\pi\kappa h}{\mu\beta}\frac{e^{\beta(p_2 - p_1)} - 1}{\ln(R_2/R_1)}\rho.}$$

Case 3: An Ideal Gas

According to Problem 4C.3, the fluid density ρ satisfies

$$\left(\frac{2\varepsilon\mu\rho_0}{\kappa}\right)\frac{\partial\rho}{\partial t} = \nabla^2\rho^2$$

in an ideal gas for porous flow. Substitute the equation of state $\rho = \rho_0 p$ to get an equation for the pressure.

$$\frac{2\varepsilon\mu\rho_0}{\kappa}\frac{\partial}{\partial t}(\rho_0 p) = \nabla^2(\rho_0 p)^2$$

$$\frac{2\varepsilon\mu\rho_0^2}{\kappa}\frac{\partial p}{\partial t} = \rho_0^2\nabla^2 p^2$$

Divide both sides by ρ_0^2 and expand the Laplacian in cylindrical coordinates by using formula (B) on page 834.

$$\frac{2\varepsilon\mu}{\kappa}\frac{\partial p}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}p^2\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}p^2 + \frac{\partial^2}{\partial z^2}p^2$$

For the radial flow in this problem, the pressure is only a function of r: $p = p(r)$. All derivatives except those with respect to r vanish as a result.

$$0 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}p^2\right)$$

Multiply both sides by r.

$$\frac{d}{dr}\left(r\frac{d}{dr}p^2\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{d}{dr}p^2 = C_5$$

Divide both sides by r.

$$\frac{d}{dr}p^2 = \frac{C_5}{r}$$

Integrate both sides with respect to r once more.

$$p^2 = C_5 \ln r + C_6$$

Apply the two boundary conditions,

$$p(R_1) = p_1$$

$$p(R_2) = p_2,$$

in order to determine C_5 and C_6.

$$p_1^2 = C_5 \ln R_1 + C_6$$

$$p_2^2 = C_5 \ln R_2 + C_6$$

Solving this system of equations yields

$$C_5 = \frac{p_2^2 - p_1^2}{\ln(R_2/R_1)} \quad \text{and} \quad C_6 = \frac{p_1^2 \ln R_2 - p_2^2 \ln R_1}{\ln(R_2/R_1)}.$$

So then

$$\begin{aligned}
p^2 &= \frac{p_2^2 - p_1^2}{\ln(R_2/R_1)} \ln r + \frac{p_1^2 \ln R_2 - p_2^2 \ln R_1}{\ln(R_2/R_1)} \\
&= \frac{p_2^2 \ln r - p_1^2 \ln r + p_1^2 \ln R_2 - p_2^2 \ln R_1}{\ln(R_2/R_1)} \\
&= \frac{p_2^2(\ln r - \ln R_1) + p_1^2(\ln R_2 - \ln r)}{\ln(R_2/R_1)} \\
&= \frac{p_2^2 \ln(r/R_1) + p_1^2 \ln(R_2/r)}{\ln(R_2/R_1)}.
\end{aligned}$$

Therefore, taking the square root of both sides,

$$\boxed{p(r) = \sqrt{\frac{p_2^2 \ln(r/R_1) + p_1^2 \ln(R_2/r)}{\ln(R_2/R_1)}}.}$$

The radial flow velocity is obtained from Darcy's equation.

$$\begin{aligned}
\mathbf{v}_0 &= -\frac{\kappa}{\mu}(\nabla p - \rho \mathbf{g}) \\
&= -\frac{\kappa}{\mu}\left(\frac{dp}{dr}\hat{\mathbf{r}} - \rho g_z \hat{\mathbf{z}}\right) \\
&= -\frac{\kappa}{\mu}\left\{\frac{1}{2}\left[\frac{p_2^2 \ln(r/R_1) + p_1^2 \ln(R_2/r)}{\ln(R_2/R_1)}\right]^{-1/2}\left[\frac{p_2^2(1/r) + p_1^2(-1/r)}{\ln(R_2/R_1)}\right]\hat{\mathbf{r}} - \rho g_z \hat{\mathbf{z}}\right\} \\
&= -\frac{\kappa}{2\mu r}\frac{p_2^2 - p_1^2}{\sqrt{[p_2^2 \ln(r/R_1) + p_1^2 \ln(R_2/r)]\ln(R_2/R_1)}}\hat{\mathbf{r}} + \frac{\kappa\rho}{\mu} g_z \hat{\mathbf{z}}
\end{aligned}$$

Therefore,

$$\boxed{v_{0r} = -\frac{\kappa}{2\mu r}\frac{p_2^2 - p_1^2}{\sqrt{[p_2^2 \ln(r/R_1) + p_1^2 \ln(R_2/r)]\ln(R_2/R_1)}}.}$$

Calculate the volumetric flow rate through a cylindrical shell of radius r, where $R_1 \le r \le R_2$.

$$\begin{aligned}
\frac{dV}{dt} &= \mathbf{v} \cdot \mathbf{A} \\
&= (v_{0r}\hat{\mathbf{r}}) \cdot (2\pi r h \hat{\mathbf{r}}) \\
&= 2\pi h r v_{0r} \\
&= -\frac{\pi\kappa h}{\mu}\frac{p_2^2 - p_1^2}{\sqrt{[p_2^2 \ln(r/R_1) + p_1^2 \ln(R_2/r)]\ln(R_2/R_1)}}
\end{aligned}$$

Multiply both sides by the fluid density ρ to get the mass flow rate through this shell.

$$\rho\frac{dV}{dt} = -\frac{\pi\kappa h}{\mu}\frac{p_2^2 - p_1^2}{\sqrt{[p_2^2 \ln(r/R_1) + p_1^2 \ln(R_2/r)]\ln(R_2/R_1)}}\rho$$

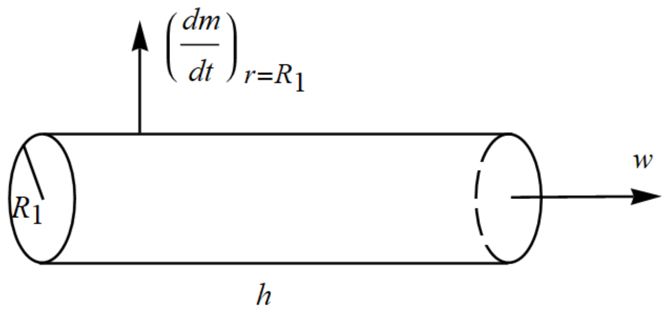
ρ is constant, so it can be brought inside the derivative.

$$\frac{d(\rho V)}{dt} = -\frac{\pi\kappa h}{\mu}\frac{p_2^2 - p_1^2}{\sqrt{[p_2^2 \ln(r/R_1) + p_1^2 \ln(R_2/r)]\ln(R_2/R_1)}}\rho$$

Density times volume is mass.

$$\frac{dm}{dt} = -\frac{\pi\kappa h}{\mu}\frac{p_2^2 - p_1^2}{\sqrt{[p_2^2 \ln(r/R_1) + p_1^2 \ln(R_2/r)] \ln(R_2/R_1)}}\rho$$

Here dm/dt represents the mass flow rate out of the shell due to the radial velocity $\mathbf{v} = v_{0r}(r)\hat{\mathbf{r}}$. Now consider the cylindrical shell of radius $r = R_1$ in particular.



Construct a mass balance over it.

$$\text{rate of mass in} - \text{rate of mass out} = \text{rate of mass accumulation}$$

Everything that flows in must flow out, so the accumulation rate is zero.

$$\text{rate of mass in} - \text{rate of mass out} = 0$$

$$0 - \left(\left.\frac{dm}{dt}\right|_{r=R_1} + w\right) = 0$$

Solve for w.

$$\begin{aligned} w &= -\left.\frac{dm}{dt}\right|_{r=R_1} \\ &= \frac{\pi\kappa h}{\mu}\frac{p_2^2 - p_1^2}{\sqrt{[p_2^2 \ln(R_1/R_1) + p_1^2 \ln(R_2/R_1)] \ln(R_2/R_1)}}\rho \\ &= \frac{\pi\kappa h}{\mu}\frac{p_2^2 - p_1^2}{\sqrt{[p_1 \ln(R_2/R_1)]^2}}\rho \end{aligned}$$

Therefore,

$$\boxed{w = \frac{\pi\kappa h}{\mu}\frac{p_2^2 - p_1^2}{p_1 \ln(R_2/R_1)}\rho.}$$