Problem 4D.1

Flow near an oscillating wall.[8] Show, by using Laplace transforms, that the complete solution to the problem stated in Eqs. 4.1-44 to 47 is

$$\frac{v_x}{v_0} = e^{-\sqrt{\omega/2\nu}y}\cos(\omega t - \sqrt{\omega/2\nu}y) - \frac{1}{\pi}\int_0^\infty e^{-\bar{\omega}t}(\sin\sqrt{\bar{\omega}/\nu}y)\frac{\bar{\omega}}{\omega^2+\bar{\omega}^2}\,d\bar{\omega} \tag{4D.1-1}$$

Solution

The equation of motion for the x-component of velocity v_x of the fluid is

$$\frac{\partial v_x}{\partial t} = \nu\frac{\partial^2 v_x}{\partial y^2}, \tag{4.1-44}$$

and the initial and boundary conditions associated with it are

I.C.:	at $t \le 0$,	$v_x = 0$	for all y	(4.1-45)
B.C. 1:	at $y = 0$,	$v_x = v_0\Re\{e^{i\omega t}\}$	for all $t > 0$	(4.1-46)
B.C. 2:	at $y = \infty$,	$v_x = 0$	for all $t > 0$,	(4.1-47)

where $\Re\{e^{i\omega t}\}$ represents the real part of oscillation, $\cos\omega t$, by Euler's formula. Physically, B.C. 1 represents a wall at $y = 0$ that is oscillating at $2\pi/\omega$ cycles per second with maximum speed v_0. Assuming that the fluid does not slip on the wall, its velocity there will be the same. Since we're interested in the solution for $t > 0$, the Laplace transform can be applied to solve the PDE. The Laplace transform of v_x is defined as

$$\mathcal{L}\{v_x(y,t)\} = \bar{v}_x(y,s) = \int_0^\infty e^{-st}v_x(y,t)\,dt.$$

As a result, the derivatives of v_x with respect to t and y transform as follows.

$$\mathcal{L}\left\{\frac{\partial v_x}{\partial t}\right\} = s\bar{v}_x - v_x(y,0)$$

$$\mathcal{L}\left\{\frac{\partial^2 v_x}{\partial y^2}\right\} = \frac{d^2\bar{v}_x}{dy^2}$$

Take the Laplace transform of both sides of the PDE

$$\mathcal{L}\left\{\frac{\partial v_x}{\partial t}\right\} = \mathcal{L}\left\{\nu\frac{\partial^2 v_x}{\partial y^2}\right\}$$

and its boundary conditions.

$$\mathcal{L}\{v_x(0,t)\} = \mathcal{L}\{v_0\cos\omega t\} \quad\rightarrow\quad \bar{v}_x(0,s) = \int_0^\infty e^{-st}v_0\cos\omega t\,dt = v_0\frac{s}{s^2+\omega^2} \tag{1}$$

$$\mathcal{L}\{v_x(\infty,t)\} = \mathcal{L}\{0\} \quad\rightarrow\quad \bar{v}_x(\infty,s) = 0 \tag{2}$$

[8] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Oxford University Press, 2nd edition (1959), p. 319, Eq. (8), with $\varepsilon = \frac{1}{2}\pi$ and $\bar{\omega} = \kappa u^2$.

Use the fact that the Laplace transform is a linear operator.

$$\mathcal{L}\left\{\frac{\partial v_x}{\partial t}\right\} = \nu\mathcal{L}\left\{\frac{\partial^2 v_x}{\partial y^2}\right\}$$

Use the formulas above to transform the derivatives.

$$s\bar{v}_x - v_x(y,0) = \nu\frac{d^2\bar{v}_x}{dy^2}$$

From the initial condition we have $v_x(y,0) = 0$. Divide both sides by ν.

$$\frac{d^2\bar{v}_x}{dy^2} = \frac{s}{\nu}\bar{v}_x$$

With the help of the Laplace transform, the PDE has been reduced to a second-order ODE whose solution can be written in terms of exponential functions.

$$\bar{v}_x(y,s) = C_1\exp\left(\sqrt{\frac{s}{\nu}}y\right) + C_2\exp\left(-\sqrt{\frac{s}{\nu}}y\right)$$

Apply the Laplace-transformed boundary conditions here to determine C_1 and C_2. For equation (2) to be satisfied, we require that $C_1 = 0$.

$$\bar{v}_x(y,\bar{\omega}) = C_2\exp\left(-\sqrt{\frac{s}{\nu}}y\right)$$

Now set $y = 0$ and use equation (1) to determine C_2.

$$\bar{v}_x(0,s) = C_2 = v_0\frac{s}{s^2+\omega^2}$$

Consequently, the Laplace-transformed solution to the problem is

$$\bar{v}_x(y,s) = v_0\frac{s}{s^2+\omega^2}\exp\left(-\sqrt{\frac{s}{\nu}}y\right).$$

Divide both sides by v_0 to make the solution non-dimensional. Also, isolate $\sqrt{s}$ in the exponential function.

$$\frac{\bar{v}_x}{v_0} = \frac{s}{s^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{s}\right)$$

The aim now is to solve for v_x/v_0 by taking the inverse Laplace transform of both sides.

$$\mathcal{L}^{-1}\left\{\frac{\bar{v}_x}{v_0}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{s}\right)\right\}$$

In order to evaluate the right side, we will resort to the definition of the inverse Laplace transform.

$$\frac{v_x}{v_0} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{s}{s^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{s}\right)ds,$$

where γ is a real constant chosen such that all singularities of the integrand lie to the left of the infinite vertical line $(\gamma - i\infty, \gamma + i\infty)$ in the complex plane.

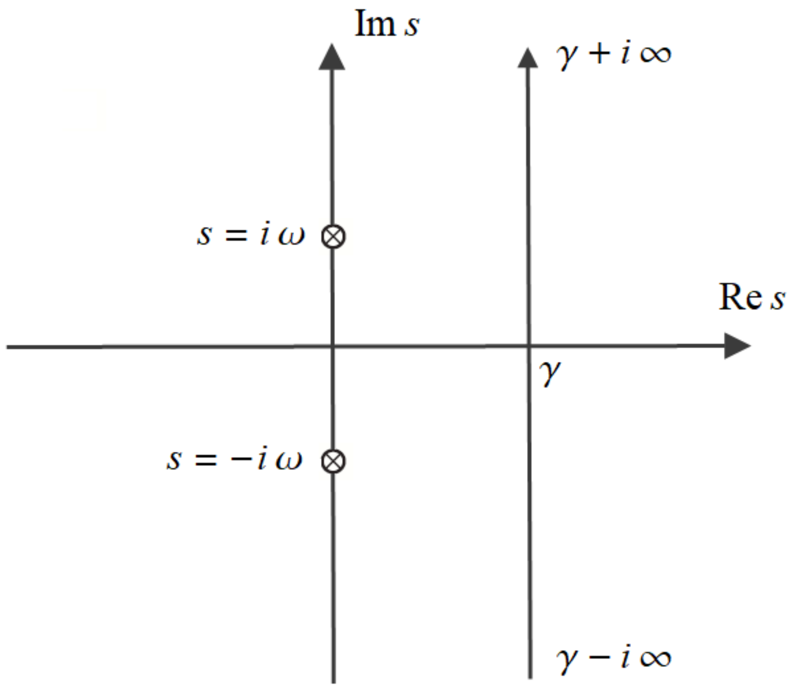

Figure 1: This is the complex plane with the singularities of the integrand marked as well as the vertical line $(\gamma - i\infty, \gamma + i\infty)$.

The integral is evaluated by considering a closed loop integral in the complex plane containing this vertical line and then applying the Cauchy residue theorem to get an equation, allowing us to solve for it. Normally the vertical line loops around back to $\gamma - i\infty$ by a semicircular arc to the left, but because of $\sqrt{\bar{\omega}}$ in the exponential function, a different path has to be taken. This is because for complex $\bar{\omega}$, the square root function can be written in terms of the logarithm.

$$\sqrt{s} = \exp\left(\frac{1}{2}\log s\right)$$

The principal branch of $\sqrt{s}$ is obtained by taking the principal branch of $\log s$.

$$\begin{aligned} &= \exp\left(\frac{1}{2}\operatorname{Log} s\right), \quad (|s| > 0,\ -\pi < \operatorname{Arg} s < \pi) \\ &= \exp\left[\frac{1}{2}(\ln r + i\Theta)\right] \\ &= \sqrt{r}e^{i\Theta/2}, \end{aligned}$$

where $r = |s|$ is the magnitude of s and $\Theta = \operatorname{Arg} s$ is the principal argument of s. In other words, this expression for $\sqrt{s}$ can be used for a complex number $s = re^{i\Theta}$. Taking the branch cut into account, the closed loop in Figure 2 will be considered.

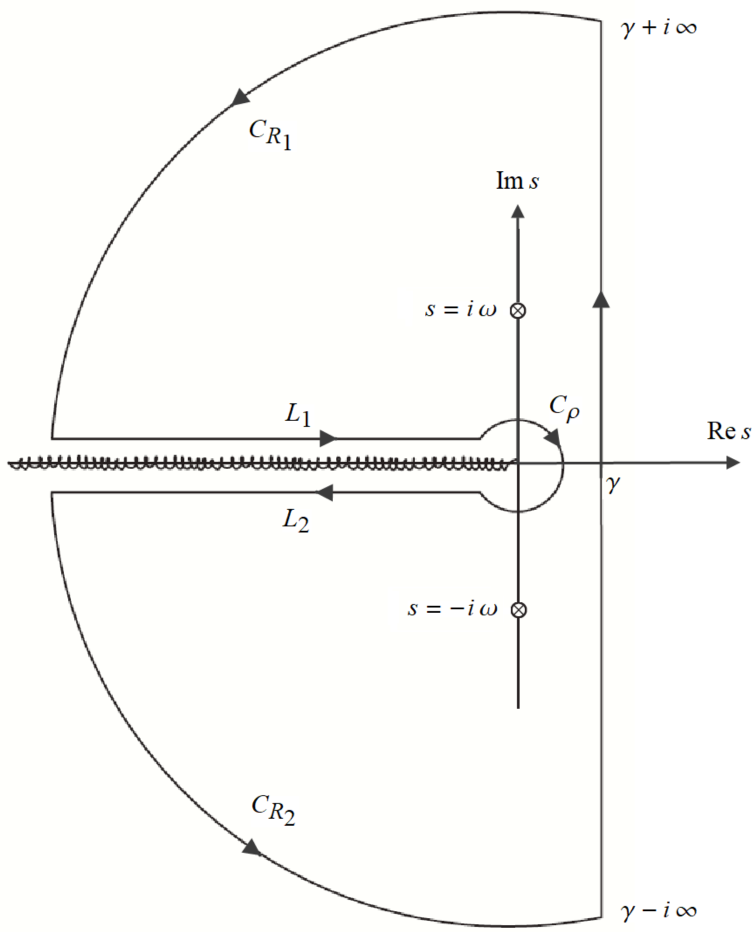

Figure 2: The branch cut ($|s| > 0$, $-\pi < \text{Arg } s < \pi$) is represented in the complex plane by the squiggly line. In order to close the integration path after traversing the vertical line, let it follow a circular arc C_{R_1}. Once the path gets to the branch cut at $\Theta = \pi$, integrate around it by going radially along L_1, around the origin by a circular arc C_ρ to the underside of the branch cut, and then radially again along L_2 at $\Theta = -\pi$. From there, let it follow a circular path C_{R_2} back to $\gamma - i\infty$.

There are two singularities enclosed in this loop, one at $s = i\omega$ and one at $s = -i\omega$. According to the Cauchy residue theorem, the closed loop integral of a function in the complex plane is equal to $2\pi i$ times the sum of the residues at the enclosed singularities. That is,

$$\oint_C e^{st}\frac{\bar{v}_x}{v_0}\,ds = 2\pi i\left(\operatorname*{Res}_{s=i\omega} e^{st}\frac{\bar{v}_x}{v_0} + \operatorname*{Res}_{s=-i\omega} e^{st}\frac{\bar{v}_x}{v_0}\right).$$

This closed loop integral is the sum of six integrals, one over each arc in the loop.

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{\bar{v}_x}{v_0}\,ds + \int_{C_{R_1}} e^{st}\frac{\bar{v}_x}{v_0}\,ds + \int_{L_1} e^{st}\frac{\bar{v}_x}{v_0}\,ds + \int_{C_\rho} e^{st}\frac{\bar{v}_x}{v_0}\,ds + \int_{L_2} e^{st}\frac{\bar{v}_x}{v_0}\,ds + \int_{C_{R_2}} e^{st}\frac{\bar{v}_x}{v_0}\,ds = 2\pi i\left(\operatorname*{Res}_{s=i\omega} e^{st}\frac{\bar{v}_x}{v_0} + \operatorname*{Res}_{s=-i\omega} e^{st}\frac{\bar{v}_x}{v_0}\right)$$

The parameterizations for the arcs are as follows.

$$\begin{aligned}
C_{R_1}:&\quad s = Re^{i\Theta}, &\quad \Theta = \frac{\pi}{2} \quad &\to \quad \Theta = \pi \\
C_{R_2}:&\quad s = Re^{i\Theta}, &\quad \Theta = -\pi \quad &\to \quad \Theta = -\frac{\pi}{2} \\
C_\rho:&\quad s = \rho e^{i\Theta}, &\quad \Theta = \pi \quad &\to \quad \Theta = -\pi \\
L_1:&\quad s = re^{i\pi}, &\quad r = R \quad &\to \quad r = \rho \\
L_2:&\quad s = re^{-i\pi}, &\quad r = \rho \quad &\to \quad r = R
\end{aligned}$$

In the limit as $R \to \infty$ the integrals over C_{R_1} and C_{R_2} vanish; also, in the limit as $\rho \to 0$ the integral over C_ρ vanishes. Proof of these statements will be given at the end. Consequently,

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{\bar{v}_x}{v_0}\,ds + \int_{L_1} e^{st}\frac{\bar{v}_x}{v_0}\,ds + \int_{L_2} e^{st}\frac{\bar{v}_x}{v_0}\,ds = 2\pi i\left(\operatorname*{Res}_{s=i\omega} e^{st}\frac{\bar{v}_x}{v_0} + \operatorname*{Res}_{s=-i\omega} e^{st}\frac{\bar{v}_x}{v_0}\right).$$

Use the parameterizations for L_1 and L_2 to obtain the sum of the integrals over these arcs.

$$\begin{aligned}
\int_{L_1} e^{st}\frac{\bar{v}_x}{v_0}\,ds + \int_{L_2} e^{st}\frac{\bar{v}_x}{v_0}\,ds &= \int_R^\rho \frac{re^{i\pi}e^{re^{i\pi}t}}{(re^{i\pi})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{r}e^{i\pi/2}\right)(e^{i\pi}\,dr) \\
&\quad + \int_\rho^R \frac{re^{-i\pi}e^{re^{-i\pi}t}}{(re^{-i\pi})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{r}e^{-i\pi/2}\right)(e^{-i\pi}\,dr)
\end{aligned}$$

Note that $e^{i\pi} = e^{-i\pi} = -1$ and $e^{i\pi/2} = i$ and $e^{-i\pi/2} = -i$.

$$\begin{aligned}
\int_{L_1} e^{st}\frac{\bar{v}_x}{v_0}\,ds + \int_{L_2} e^{st}\frac{\bar{v}_x}{v_0}\,ds &= \int_R^\rho \frac{(-r)e^{(-r)t}}{(-r)^2+\omega^2}\exp\left[-\frac{y}{\sqrt{\nu}}\sqrt{r}(i)\right](-dr) \\
&\quad + \int_\rho^R \frac{(-r)e^{(-r)t}}{(-r)^2+\omega^2}\exp\left[-\frac{y}{\sqrt{\nu}}\sqrt{r}(-i)\right](-dr)
\end{aligned}$$

$$\begin{aligned}
\int_{L_1} e^{st}\frac{\bar{v}_x}{v_0}\,ds + \int_{L_2} e^{st}\frac{\bar{v}_x}{v_0}\,ds &= \int_R^\rho \frac{re^{-rt}}{r^2+\omega^2}\exp\left(-i\frac{y}{\sqrt{\nu}}\sqrt{r}\right)dr + \int_\rho^R \frac{re^{-rt}}{r^2+\omega^2}\exp\left(i\frac{y}{\sqrt{\nu}}\sqrt{r}\right)dr \\
&= -\int_\rho^R \frac{re^{-rt}}{r^2+\omega^2}\exp\left(-i\frac{y}{\sqrt{\nu}}\sqrt{r}\right)dr + \int_\rho^R \frac{re^{-rt}}{r^2+\omega^2}\exp\left(i\frac{y}{\sqrt{\nu}}\sqrt{r}\right)dr \\
&= \int_\rho^R \frac{re^{-rt}}{r^2+\omega^2}\left[\exp\left(i\frac{y}{\sqrt{\nu}}\sqrt{r}\right) - \exp\left(-i\frac{y}{\sqrt{\nu}}\sqrt{r}\right)\right]dr \\
&= 2i\int_\rho^R \frac{re^{-rt}}{r^2+\omega^2}\,\frac{\exp\left(i\frac{y}{\sqrt{\nu}}\sqrt{r}\right) - \exp\left(-i\frac{y}{\sqrt{\nu}}\sqrt{r}\right)}{2i}\,dr \\
&= 2i\int_\rho^R \frac{re^{-rt}}{r^2+\omega^2}\sin\left(\frac{y}{\sqrt{\nu}}\sqrt{r}\right)dr
\end{aligned}$$

In the limit as $R \to \infty$ and $\rho \to 0$, we get

$$\int_{L_1} e^{st}\frac{\bar{v}_x}{v_0}\,ds + \int_{L_2} e^{st}\frac{\bar{v}_x}{v_0}\,ds = 2i\int_0^\infty \frac{re^{-rt}}{r^2+\omega^2}\sin\left(\sqrt{\frac{r}{\nu}}y\right)dr.$$

Thus, Cauchy's residue theorem becomes

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{\bar{v}_x}{v_0}\,ds + 2i\int_0^\infty \frac{re^{-rt}}{r^2+\omega^2}\sin\left(\sqrt{\frac{r}{\nu}}y\right)dr = 2\pi i\left(\operatorname*{Res}_{s=i\omega} e^{st}\frac{\bar{v}_x}{v_0} + \operatorname*{Res}_{s=-i\omega} e^{st}\frac{\bar{v}_x}{v_0}\right).$$

The next step is to evaluate the residues at the enclosed singularities.

$$\begin{aligned}
\operatorname*{Res}_{s=i\omega} e^{st}\frac{\bar{v}_x}{v_0} &= \operatorname*{Res}_{s=i\omega} \frac{se^{st}}{s^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{s}\right) = \operatorname*{Res}_{s=i\omega}\frac{\phi_1(s)}{s-i\omega} = \phi_1(i\omega)\\
\operatorname*{Res}_{s=-i\omega} e^{st}\frac{\bar{v}_x}{v_0} &= \operatorname*{Res}_{s=-i\omega} \frac{se^{st}}{s^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{s}\right) = \operatorname*{Res}_{s=-i\omega}\frac{\phi_2(s)}{s+i\omega} = \phi_2(-i\omega)
\end{aligned}$$

Because $s = i\omega$ and $s = -i\omega$ are simple poles, the residues there are $\phi_1(i\omega)$ and $\phi_2(-i\omega)$, respectively.

$$\begin{aligned}
\phi_1(s) &= \frac{se^{st}}{s+i\omega}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{s}\right) \quad\rightarrow\quad \phi_1(i\omega) = \frac{i\omega e^{i\omega t}}{2i\omega}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{\omega}e^{i\pi/4}\right)\\
\phi_2(s) &= \frac{se^{st}}{s-i\omega}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{s}\right) \quad\rightarrow\quad \phi_2(-i\omega) = \frac{-i\omega e^{-i\omega t}}{-2i\omega}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{\omega}e^{-i\pi/4}\right)
\end{aligned}$$

Apply Euler's formula to simplify the formulas for $\phi_1(i\omega)$ and $\phi_2(-i\omega)$.

$$\begin{aligned}
\phi_1(i\omega) &= \frac{1}{2}e^{i\omega t}\exp\left[-\sqrt{\frac{\omega}{\nu}}y\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right] = \frac{1}{2}e^{i\omega t}\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\exp\left(-i\sqrt{\frac{\omega}{2\nu}}y\right)\\
\phi_2(-i\omega) &= \frac{1}{2}e^{-i\omega t}\exp\left[-\sqrt{\frac{\omega}{\nu}}y\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\right] = \frac{1}{2}e^{-i\omega t}\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\exp\left(i\sqrt{\frac{\omega}{2\nu}}y\right)
\end{aligned}$$

Combine the complex exponential functions.

$$\begin{aligned}
\phi_1(i\omega) &= \frac{1}{2}\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\exp\left[i\left(\omega t-\sqrt{\frac{\omega}{2\nu}}y\right)\right]\\
\phi_2(-i\omega) &= \frac{1}{2}\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\exp\left[-i\left(\omega t-\sqrt{\frac{\omega}{2\nu}}y\right)\right]
\end{aligned}$$

As a result, the sum of the residues is

$$\begin{aligned}
\operatorname*{Res}_{s=i\omega} e^{st}\frac{\bar{v}_x}{v_0} + \operatorname*{Res}_{s=-i\omega} e^{st}\frac{\bar{v}_x}{v_0} &= \frac{1}{2}\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\exp\left[i\left(\omega t-\sqrt{\frac{\omega}{2\nu}}y\right)\right] + \frac{1}{2}\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\exp\left[-i\left(\omega t-\sqrt{\frac{\omega}{2\nu}}y\right)\right]\\
&= \exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\frac{\exp\left[i\left(\omega t-\sqrt{\frac{\omega}{2\nu}}y\right)\right]+\exp\left[-i\left(\omega t-\sqrt{\frac{\omega}{2\nu}}y\right)\right]}{2}\\
&= \exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\cos\left(\omega t-\sqrt{\frac{\omega}{2\nu}}y\right).
\end{aligned}$$

Thus, Cauchy's residue theorem becomes

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{\bar{v}_x}{v_0}\,ds + 2i\int_0^\infty \frac{re^{-rt}}{r^2+\omega^2}\sin\left(\sqrt{\frac{r}{\nu}}y\right)dr = 2\pi i\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\cos\left(\omega t-\sqrt{\frac{\omega}{2\nu}}y\right).$$

Move the improper integral over to the right side.

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{\bar{v}_x}{v_0}\,ds = 2\pi i\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\cos\left(\omega t-\sqrt{\frac{\omega}{2\nu}}y\right) - 2i\int_0^\infty \frac{re^{-rt}}{r^2+\omega^2}\sin\left(\sqrt{\frac{r}{\nu}}y\right)dr$$

Divide both sides by $2\pi i$.

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{\bar{v}_x}{v_0}\,ds = \exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right) - \frac{1}{\pi}\int_0^\infty \frac{re^{-rt}}{r^2+\omega^2}\sin\left(\sqrt{\frac{r}{\nu}}y\right)dr$$

Let $r = \bar{\omega}$. Therefore,

$$\boxed{\frac{v_x}{v_0} = \exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right) - \frac{1}{\pi}\int_0^\infty e^{-\bar{\omega}t}\sin\left(\sqrt{\frac{\bar{\omega}}{\nu}}y\right)\frac{\bar{\omega}}{\bar{\omega}^2+\omega^2}\,d\bar{\omega}, \quad t>0.}$$

The term with the improper integral is the transient part of the solution. As t gets big, the integral makes less and less of a contribution to the fluid velocity. The other term is the steady-state solution, that is, the solution for large t. Notice that the velocity falls off exponentially with distance from the wall y. Also, there is a phase shift in the cosine's argument, indicating that there is lag. When the wall moves, it takes time for the fluid some distance away to feel that motion. This time is proportional to that distance.

The Integral Over C_ρ

The aim here is to show that the integral over C_ρ tends to zero as $\rho \to 0$. Recall that the parameterization on C_ρ is $s = \rho e^{i\Theta}$, where Θ goes from π to $-\pi$.

$$\begin{aligned}
\int_{C_\rho} e^{st}\frac{\bar{v}_x}{v_0}\,ds &= \int_{C_\rho} \frac{se^{st}}{s^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{s}\right)ds \\
&= \int_\pi^{-\pi} \frac{\rho e^{i\Theta}e^{\rho e^{i\Theta}t}}{(\rho e^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{\rho}e^{i\Theta/2}\right)(i\rho e^{i\Theta}\,d\Theta) \\
&= i\int_\pi^{-\pi} \frac{\rho^2 e^{i2\Theta}e^{\rho e^{i\Theta}t}}{(\rho e^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{\rho}e^{i\Theta/2}\right)d\Theta
\end{aligned}$$

Now take the limit as $\rho \to 0$.

$$\lim_{\rho\to 0}\int_{C_\rho} e^{st}\frac{\bar{v}_x}{v_0}\,ds = \lim_{\rho\to 0} i\int_\pi^{-\pi} \frac{\rho^2 e^{i2\Theta}e^{\rho e^{i\Theta}t}}{(\rho e^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{\rho}e^{i\Theta/2}\right)d\Theta$$

Because the limits of integration don't depend on ρ, the limit can be brought inside the integral.

$$= i\int_\pi^{-\pi} \lim_{\rho\to 0}\frac{\rho^2 e^{i2\Theta}e^{\rho e^{i\Theta}t}}{(\rho e^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{\rho}e^{i\Theta/2}\right)d\Theta$$

The integrand tends to 0 because of ρ^2 in the numerator. Therefore,

$$\lim_{\rho\to 0}\int_{C_\rho} e^{st}\frac{\bar{v}_x}{v_0}\,ds = 0.$$

The Integral Over C_{R_1}

Here we will show that the integral over C_{R_1} tends to zero as $R \to \infty$. Recall that the parameterization on C_{R_1} is $s = Re^{i\Theta}$, where Θ goes from $\frac{\pi}{2}$ to π.

$$\begin{aligned}\int_{C_{R_1}} e^{st}\frac{\bar{v}_x}{v_0}\,ds &= \int_{C_{R_1}} \frac{se^{st}}{s^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{s}\right)ds \\ &= \int_{\frac{\pi}{2}}^{\pi} \frac{Re^{i\Theta}e^{Re^{i\Theta}t}}{(Re^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}e^{i\Theta/2}\right)(iRe^{i\Theta}\,d\Theta) \\ &= i\int_{\frac{\pi}{2}}^{\pi} \frac{R^2e^{i2\Theta}e^{Re^{i\Theta}t}}{(Re^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}e^{i\Theta/2}\right)d\Theta\end{aligned}$$

Apply Euler's formula to the complex exponential functions inside the exponential functions.

$$\begin{aligned}&= i\int_{\frac{\pi}{2}}^{\pi} \frac{R^2e^{i2\Theta}e^{Rt(\cos\Theta+i\sin\Theta)}}{(Re^{i\Theta})^2+\omega^2}\exp\left[-\frac{y}{\sqrt{\nu}}\sqrt{R}\left(\cos\frac{\Theta}{2}+i\sin\frac{\Theta}{2}\right)\right]d\Theta \\ &= i\int_{\frac{\pi}{2}}^{\pi} \frac{R^2e^{i2\Theta}e^{Rt\cos\Theta}e^{iRt\sin\Theta}}{(Re^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)\exp\left(-i\frac{y}{\sqrt{\nu}}\sqrt{R}\sin\frac{\Theta}{2}\right)d\Theta\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_{R_1}} e^{st}\frac{\bar{v}_x}{v_0}\,ds\right| &= \left|i\int_{\frac{\pi}{2}}^{\pi} \frac{R^2e^{i2\Theta}e^{Rt\cos\Theta}e^{iRt\sin\Theta}}{(Re^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)\exp\left(-i\frac{y}{\sqrt{\nu}}\sqrt{R}\sin\frac{\Theta}{2}\right)d\Theta\right| \\ &\le \int_{\frac{\pi}{2}}^{\pi}\left|\frac{R^2e^{i2\Theta}e^{Rt\cos\Theta}e^{iRt\sin\Theta}}{(Re^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)\exp\left(-i\frac{y}{\sqrt{\nu}}\sqrt{R}\sin\frac{\Theta}{2}\right)\right|d\Theta \\ &= \int_{\frac{\pi}{2}}^{\pi}\frac{|R^2||e^{i2\Theta}||e^{Rt\cos\Theta}||e^{iRt\sin\Theta}|}{|(Re^{i\Theta})^2+\omega^2|}\left|\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)\right|\left|\exp\left(-i\frac{y}{\sqrt{\nu}}\sqrt{R}\sin\frac{\Theta}{2}\right)\right|d\Theta\end{aligned}$$

The complex exponential functions never have a magnitude higher than 1. Also, for two complex numbers, z_1 and z_2, $|z_1+z_2| \ge ||z_1|-|z_2||$. For the denominator then,
$|(Re^{i\Theta})^2+\omega^2| \ge ||(Re^{i\Theta})^2|-|\omega^2|| = ||Re^{i\Theta}|^2-|\omega^2|| = R^2-\omega^2$.

$$\le \int_{\frac{\pi}{2}}^{\pi}\frac{R^2e^{Rt\cos\Theta}}{R^2-\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)d\Theta$$

Divide the numerator and denominator by R^2.

$$\le \int_{\frac{\pi}{2}}^{\pi}\frac{e^{Rt\cos\Theta}}{1-\frac{\omega^2}{R^2}}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)d\Theta$$

Take the limit as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_{R_1}} e^{st}\frac{\bar{v}_x}{v_0}\,ds\right| \le \lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\frac{e^{Rt\cos\Theta}}{1-\frac{\omega^2}{R^2}}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)d\Theta$$

The limits of integration do not depend on R, so the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_{R_1}} e^{st}\frac{\bar{v}_x}{v_0}\,ds\right| \leq \int_{\frac{\pi}{2}}^{\pi} \lim_{R\to\infty} \frac{e^{Rt\cos\Theta}}{1-\frac{\omega^2}{R^2}} \exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right) d\Theta$$

The denominator goes to 1. For $\frac{\pi}{2} < \Theta < \pi$, $\cos\Theta$ is negative. On the other hand, both $t > 0$ and $R > 0$, so

$$e^{Rt\cos\Theta} \to 0 \quad \text{as} \quad R \to \infty.$$

In the second exponential function $y > 0$, $\nu > 0$, and $R > 0$. For $\frac{\pi}{4} < \frac{\Theta}{2} < \frac{\pi}{2}$, $\cos\frac{\Theta}{2}$ is positive, so

$$\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right) \to 0 \quad \text{as} \quad R \to \infty.$$

Thus, the right side is zero.

$$\lim_{R\to\infty}\left|\int_{C_{R_1}} e^{st}\frac{\bar{v}_x}{v_0}\,ds\right| \leq 0$$

Magnitudes cannot be negative, so the limit must be equal to zero.

$$\lim_{R\to\infty}\left|\int_{C_{R_1}} e^{st}\frac{\bar{v}_x}{v_0}\,ds\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_{R_1}} e^{st}\frac{\bar{v}_x}{v_0}\,ds = 0.$$

The Integral Over C_{R_2}

Here we will show that the integral over C_{R_2} tends to zero as $R \to \infty$. Recall that the parameterization on C_{R_2} is $s = Re^{i\Theta}$, where Θ goes from $-\pi$ to $-\frac{\pi}{2}$.

$$\begin{aligned}
\int_{C_{R_2}} e^{st}\frac{\bar{v}_x}{v_0}\,ds &= \int_{C_{R_2}} \frac{se^{st}}{s^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{s}\right)ds \\
&= \int_{-\pi}^{-\frac{\pi}{2}} \frac{Re^{i\Theta}e^{Re^{i\Theta}t}}{(Re^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}e^{i\Theta/2}\right)(iRe^{i\Theta}\,d\Theta) \\
&= i\int_{-\pi}^{-\frac{\pi}{2}} \frac{R^2e^{i2\Theta}e^{Re^{i\Theta}t}}{(Re^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}e^{i\Theta/2}\right)d\Theta
\end{aligned}$$

Apply Euler's formula to the complex exponential functions inside the exponential functions.

$$\begin{aligned}
&= i\int_{-\pi}^{-\frac{\pi}{2}} \frac{R^2e^{i2\Theta}e^{Rt(\cos\Theta+i\sin\Theta)}}{(Re^{i\Theta})^2+\omega^2}\exp\left[-\frac{y}{\sqrt{\nu}}\sqrt{R}\left(\cos\frac{\Theta}{2}+i\sin\frac{\Theta}{2}\right)\right]d\Theta \\
&= i\int_{-\pi}^{-\frac{\pi}{2}} \frac{R^2e^{i2\Theta}e^{Rt\cos\Theta}e^{iRt\sin\Theta}}{(Re^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)\exp\left(-i\frac{y}{\sqrt{\nu}}\sqrt{R}\sin\frac{\Theta}{2}\right)d\Theta
\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}
\left|\int_{C_{R_2}} e^{st}\frac{\bar{v}_x}{v_0}\,ds\right| &= \left|i\int_{-\pi}^{-\frac{\pi}{2}} \frac{R^2e^{i2\Theta}e^{Rt\cos\Theta}e^{iRt\sin\Theta}}{(Re^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)\exp\left(-i\frac{y}{\sqrt{\nu}}\sqrt{R}\sin\frac{\Theta}{2}\right)d\Theta\right| \\
&\le \int_{-\pi}^{-\frac{\pi}{2}} \left|\frac{R^2e^{i2\Theta}e^{Rt\cos\Theta}e^{iRt\sin\Theta}}{(Re^{i\Theta})^2+\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)\exp\left(-i\frac{y}{\sqrt{\nu}}\sqrt{R}\sin\frac{\Theta}{2}\right)\right|d\Theta \\
&= \int_{-\pi}^{-\frac{\pi}{2}} \frac{|R^2||e^{i2\Theta}||e^{Rt\cos\Theta}||e^{iRt\sin\Theta}|}{|(Re^{i\Theta})^2+\omega^2|}\left|\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)\right|\left|\exp\left(-i\frac{y}{\sqrt{\nu}}\sqrt{R}\sin\frac{\Theta}{2}\right)\right|d\Theta
\end{aligned}$$

The complex exponential functions never have a magnitude higher than 1. Also, for two complex numbers, z_1 and z_2, $|z_1+z_2| \ge ||z_1|-|z_2||$. For the denominator then,
$|(Re^{i\Theta})^2+\omega^2| \ge ||(Re^{i\Theta})^2|-|\omega^2|| = ||Re^{i\Theta}|^2-|\omega^2|| = R^2-\omega^2$.

$$\le \int_{-\pi}^{-\frac{\pi}{2}} \frac{R^2e^{Rt\cos\Theta}}{R^2-\omega^2}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)d\Theta$$

Divide the numerator and denominator by R^2.

$$\le \int_{-\pi}^{-\frac{\pi}{2}} \frac{e^{Rt\cos\Theta}}{1-\frac{\omega^2}{R^2}}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)d\Theta$$

Take the limit as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_{R_2}} e^{st}\frac{\bar{v}_x}{v_0}\,ds\right| \le \lim_{R\to\infty}\int_{-\pi}^{-\frac{\pi}{2}} \frac{e^{Rt\cos\Theta}}{1-\frac{\omega^2}{R^2}}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)d\Theta$$

The limits of integration do not depend on R, so the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_{R_2}} e^{st}\frac{\bar{v}_x}{v_0}\,ds\right| \leq \int_{-\pi}^{-\frac{\pi}{2}} \lim_{R\to\infty}\frac{e^{Rt\cos\Theta}}{1-\frac{\omega^2}{R^2}}\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right)d\Theta$$

The denominator goes to 1. For $-\pi < \Theta < -\frac{\pi}{2}$, $\cos\Theta$ is negative. On the other hand, both $t > 0$ and $R > 0$, so

$$e^{Rt\cos\Theta} \to 0 \quad \text{as} \quad R \to \infty.$$

In the second exponential function $y > 0$, $\nu > 0$, and $R > 0$. For $-\frac{\pi}{2} < \frac{\Theta}{2} < -\frac{\pi}{4}$, $\cos\frac{\Theta}{2}$ is positive, so

$$\exp\left(-\frac{y}{\sqrt{\nu}}\sqrt{R}\cos\frac{\Theta}{2}\right) \to 0 \quad \text{as} \quad R \to \infty.$$

Thus, the right side is zero.

$$\lim_{R\to\infty}\left|\int_{C_{R_2}} e^{st}\frac{\bar{v}_x}{v_0}\,ds\right| \leq 0$$

Magnitudes cannot be negative, so the limit must be equal to zero.

$$\lim_{R\to\infty}\left|\int_{C_{R_2}} e^{st}\frac{\bar{v}_x}{v_0}\,ds\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_{R_2}} e^{st}\frac{\bar{v}_x}{v_0}\,ds = 0.$$