Problem 4C.3

Darcy's equation for flow through porous media. For the flow of a fluid through a porous medium, the equations of continuity and motion may be replaced by

smoothed continuity equation $$\varepsilon \frac{\partial \rho}{\partial t} = -(\nabla \cdot \rho \mathbf{v}_0) \tag{4C.3-1}$$

Darcy's equation[5] $$\mathbf{v}_0 = -\frac{\kappa}{\mu}(\nabla p - \rho \mathbf{g}) \tag{4C.3-2}$$

in which ε, the *porosity*, is the ratio of pore volume to total volume, and κ is the *permeability* of the porous medium. The velocity $\mathbf{v}_0$ in these equations is the *superficial velocity*, which is defined as the volume rate of flow through a unit cross-sectional area of the solid plus fluid, averaged over a small region of space—small with respect to the macroscopic dimensions in the flow system, but large with respect to the pore size. The density and pressure are averaged over a region available to flow that is large with respect to the pore size. Equation 4C.3-2 was proposed empirically to describe the slow seepage of fluids through granular media.

When Eqs. 4C.3-1 and 2 are combined we get

$$\left(\frac{\varepsilon\mu}{\kappa}\right)\frac{\partial \rho}{\partial t} = (\nabla \cdot \rho(\nabla p - \rho \mathbf{g})) \tag{4C.3-3}$$

for constant viscosity and permeability. This equation and the equation of state describe the motion of a fluid in a porous medium. For most purposes we may write the *equation of state* as

$$\rho = \rho_0 p^m e^{\beta p} \tag{4C.3-4}$$

in which ρ_0 is the fluid density at unit pressure, and the following parameters have been given:[6]

1. Incompressible liquids $m = 0$ $\beta = 0$
2. Compressible liquids $m = 0$ $\beta \neq 0$
3. Isothermal expansion of gases $\beta = 0$ $m = 1$
4. Adiabatic expansion of gases $\beta = 0$ $m = C_V/C_p = 1/\gamma$

Show that Eqs. 4C.3-3 and 4 can be combined and simplified for these four categories to give (for gases it is customary to neglect the gravity terms since they are small compared with the pressure terms):

Case 1. $$\nabla^2 \mathscr{P} = 0 \tag{4C.3-5}$$

Case 2. $$\left(\frac{\varepsilon\mu\beta}{\kappa}\right)\frac{\partial \rho}{\partial t} = \nabla^2 \rho - (\nabla \cdot \rho^2 \beta \mathbf{g}) \tag{4C.3-6}$$

Case 3. $$\left(\frac{2\varepsilon\mu\rho_0}{\kappa}\right)\frac{\partial \rho}{\partial t} = \nabla^2 \rho^2 \tag{4C.3-7}$$

Case 4. $$\left(\frac{(m+1)\varepsilon\mu\rho_0^{1/m}}{\kappa}\right)\frac{\partial \rho}{\partial t} = \nabla^2 \rho^{(1+m)/m} \tag{4C.3-8}$$

[5]**Henry Philibert Gaspard Darcy** (1803–1858) studied in Paris and became famous for designing the municipal water-supply system in Dijon, the city of his birth. H. Darcy, *Les Fontaines Publiques de la Ville de Dijon*, Victor Dalmont, Paris (1856). For further discussion of "Darcy's law," see J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics*, Martinus Nihjoff, Dordrecht (1983); and H. Brenner and D. A. Edwards, *Macrotransport Processes*, Butterworth-Heinemann, Boston (1993).

Note that Case 1 leads to *Laplace's equation*, Case 2 without the gravity term leads to the *heat conduction* or *diffusion equation*, and Cases 3 and 4 lead to nonlinear equations.[7]

Solution

Case 1: Incompressible Liquids ($m = 0$ and $\beta = 0$)

If $m = 0$ and $\beta = 0$, then the equation of state simplifies to $\rho = \rho_0$. Substitute $\rho = \rho_0$ into both sides of Eq. 4C.3-3.

$$\left(\frac{\varepsilon\mu}{\kappa}\right)\frac{\partial}{\partial t}(\rho_0) = \nabla\cdot\rho_0(\nabla p - \rho_0\mathbf{g}).$$

Since ρ_0 is a constant, the left side is zero and ρ_0 can be pulled in front of ∇ on the right side.

$$0 = \rho_0\nabla\cdot(\nabla p - \rho_0\mathbf{g})$$

Divide both sides by ρ_0 and let the modified pressure $\mathscr{P}$ be defined by $\nabla\mathscr{P} = \nabla p - \rho_0\mathbf{g}$.

$$0 = \nabla\cdot\nabla\mathscr{P}$$

Therefore,

$$\nabla^2\mathscr{P} = 0.$$

Case 2: Compressible Liquids ($m = 0$ and $\beta \neq 0$)

If $m = 0$ and $\beta \neq 0$, then the equation of state simplifies to $\rho = \rho_0 e^{\beta p}$. Substitute $\rho = \rho_0 e^{\beta p}$ into the right side of Eq. 4C.3-3.

$$\begin{aligned}
\left(\frac{\varepsilon\mu}{\kappa}\right)\frac{\partial\rho}{\partial t} &= \nabla\cdot\rho_0 e^{\beta p}(\nabla p - \rho_0 e^{\beta p}\mathbf{g}) \\
&= \nabla\cdot(\rho_0 e^{\beta p}\nabla p - \rho_0^2 e^{2\beta p}\mathbf{g}) \\
&= \nabla\cdot\left(\rho_0 e^{\beta p}\left\langle\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right\rangle - \rho^2\mathbf{g}\right) \\
&= \nabla\cdot\left(\left\langle\rho_0 e^{\beta p}\frac{\partial p}{\partial x}, \rho_0 e^{\beta p}\frac{\partial p}{\partial y}, \rho_0 e^{\beta p}\frac{\partial p}{\partial z}\right\rangle - \rho^2\mathbf{g}\right) \\
&= \nabla\cdot\left[\left\langle\frac{1}{\beta}\frac{\partial}{\partial x}(\rho_0 e^{\beta p}), \frac{1}{\beta}\frac{\partial}{\partial y}(\rho_0 e^{\beta p}), \frac{1}{\beta}\frac{\partial}{\partial z}(\rho_0 e^{\beta p})\right\rangle - \rho^2\mathbf{g}\right] \\
&= \nabla\cdot\left[\frac{1}{\beta}\left\langle\frac{\partial}{\partial x}(\rho), \frac{\partial}{\partial y}(\rho), \frac{\partial}{\partial z}(\rho)\right\rangle - \rho^2\mathbf{g}\right] \\
&= \nabla\cdot\left(\frac{1}{\beta}\nabla\rho - \rho^2\mathbf{g}\right) \\
&= \frac{1}{\beta}\nabla\cdot\nabla\rho - \nabla\cdot\rho^2\mathbf{g}
\end{aligned}$$

Therefore, multiplying both sides by β,

$$\left(\frac{\varepsilon\mu\beta}{\kappa}\right)\frac{\partial\rho}{\partial t} = \nabla^2\rho - \nabla\cdot(\rho^2\beta\mathbf{g}).$$

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[6] M. Muskat, *Flow of Homogeneous Fluids Through Porous Media*, McGraw-Hill (1937).

[7] For the boundary condition at a porous surface that bounds a moving fluid, see G. S. Beavers and D. D. Joseph, *J. Fluid Mech.*, **30**, 197-207 (1967) and G. S. Beavers, E. M. Sparrow, and B. A. Masha, *AIChE Journal*, **20**, 596-597 (1974).

Case 3: Isothermal Expansion of Gases ($m = 1$ and $\beta = 0$)

Neglecting the gravity term (justified for gases), Eq. 4C.3-3 reduces to

$$\left(\frac{\varepsilon\mu}{\kappa}\right)\frac{\partial\rho}{\partial t} = \nabla \cdot \rho\nabla p.$$

If $m = 1$ and $\beta = 0$, then the equation of state simplifies to $\rho = \rho_0 p$. Substitute $\rho = \rho_0 p$ into the right side.

$$\begin{aligned}
\left(\frac{\varepsilon\mu}{\kappa}\right)\frac{\partial\rho}{\partial t} &= \nabla \cdot \rho_0 p \nabla p \\
&= \frac{1}{\rho_0}\nabla \cdot \rho_0^2 p \left\langle \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right\rangle \\
&= \frac{1}{\rho_0}\nabla \cdot \left\langle \rho_0^2 p \frac{\partial p}{\partial x}, \rho_0^2 p \frac{\partial p}{\partial y}, \rho_0^2 p \frac{\partial p}{\partial z} \right\rangle \\
&= \frac{1}{\rho_0}\nabla \cdot \left\langle \frac{1}{2}\frac{\partial}{\partial x}(\rho_0^2 p^2), \frac{1}{2}\frac{\partial}{\partial y}(\rho_0^2 p^2), \frac{1}{2}\frac{\partial}{\partial z}(\rho_0^2 p^2) \right\rangle \\
&= \frac{1}{2\rho_0}\nabla \cdot \left\langle \frac{\partial}{\partial x}(\rho^2), \frac{\partial}{\partial y}(\rho^2), \frac{\partial}{\partial z}(\rho^2) \right\rangle \\
&= \frac{1}{2\rho_0}\nabla \cdot \nabla\rho^2
\end{aligned}$$

Therefore, multiplying both sides by $2\rho_0$,

$$\left(\frac{2\varepsilon\mu\rho_0}{\kappa}\right)\frac{\partial\rho}{\partial t} = \nabla^2\rho^2.$$

Case 4: Adiabatic Expansion of Gases ($m \neq 0$ and $\beta = 0$)

Neglecting the gravity term (justified for gases), Eq. 4C.3-3 reduces to

$$\left(\frac{\varepsilon\mu}{\kappa}\right)\frac{\partial\rho}{\partial t} = \nabla\cdot\rho\nabla p.$$

If $\beta = 0$, then the equation of state simplifies to $\rho = \rho_0 p^m$. Substitute $\rho = \rho_0 p^m$ into the right side.

$$\begin{aligned}
\left(\frac{\varepsilon\mu}{\kappa}\right)\frac{\partial\rho}{\partial t} &= \nabla\cdot\rho_0 p^m\nabla p \\
&= \rho_0\nabla\cdot p^m\left\langle\frac{\partial p}{\partial x},\frac{\partial p}{\partial y},\frac{\partial p}{\partial z}\right\rangle \\
&= \rho_0\nabla\cdot\left\langle p^m\frac{\partial p}{\partial x},p^m\frac{\partial p}{\partial y},p^m\frac{\partial p}{\partial z}\right\rangle \\
&= \rho_0\nabla\cdot\left\langle\frac{1}{m+1}\frac{\partial}{\partial x}(p^{m+1}),\frac{1}{m+1}\frac{\partial}{\partial y}(p^{m+1}),\frac{1}{m+1}\frac{\partial}{\partial z}(p^{m+1})\right\rangle \\
&= \frac{\rho_0}{m+1}\nabla\cdot\left\langle\frac{\partial}{\partial x}(p^m\cdot p),\frac{\partial}{\partial y}(p^m\cdot p),\frac{\partial}{\partial z}(p^m\cdot p)\right\rangle \\
&= \frac{\rho_0}{m+1}\nabla\cdot\left\langle\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot\sqrt[m]{\frac{\rho}{\rho_0}}\right),\frac{\partial}{\partial y}\left(\frac{\rho}{\rho_0}\cdot\sqrt[m]{\frac{\rho}{\rho_0}}\right),\frac{\partial}{\partial z}\left(\frac{\rho}{\rho_0}\cdot\sqrt[m]{\frac{\rho}{\rho_0}}\right)\right\rangle \\
&= \frac{1}{m+1}\nabla\cdot\left\langle\frac{\partial}{\partial x}\left(\frac{\rho^{1+1/m}}{\rho_0^{1/m}}\right),\frac{\partial}{\partial y}\left(\frac{\rho^{1+1/m}}{\rho_0^{1/m}}\right),\frac{\partial}{\partial z}\left(\frac{\rho^{1+1/m}}{\rho_0^{1/m}}\right)\right\rangle \\
&= \frac{1}{(m+1)\rho_0^{1/m}}\nabla\cdot\left\langle\frac{\partial}{\partial x}[\rho^{(m+1)/m}],\frac{\partial}{\partial y}[\rho^{(m+1)/m}],\frac{\partial}{\partial z}[\rho^{(m+1)/m}]\right\rangle \\
&= \frac{1}{(m+1)\rho_0^{1/m}}\nabla\cdot\nabla\rho^{(m+1)/m}
\end{aligned}$$

Therefore, multiplying both sides by $(m+1)\rho_0^{1/m}$,

$$\left(\frac{(m+1)\varepsilon\mu\rho_0^{1/m}}{\kappa}\right)\frac{\partial\rho}{\partial t} = \nabla^2\rho^{(1+m)/m}.$$