Problem 4C.3

Darcy's equation for flow through porous media. For the flow of a fluid through a porous medium, the equations of continuity and motion may be replaced by

smoothed continuity equation $$\varepsilon \frac{\partial \rho}{\partial t} = -(\nabla \cdot \rho \mathbf{v}_0) \tag{4C.3-1}$$

Darcy's equation[5] $$\mathbf{v}_0 = -\frac{\kappa}{\mu}(\nabla p - \rho \mathbf{g}) \tag{4C.3-2}$$

in which ε, the *porosity*, is the ratio of pore volume to total volume, and κ is the *permeability* of the porous medium. The velocity $\mathbf{v}_0$ in these equations is the *superficial velocity*, which is defined as the volume rate of flow through a unit cross-sectional area of the solid plus fluid, averaged over a small region of space—small with respect to the macroscopic dimensions in the flow system, but large with respect to the pore size. The density and pressure are averaged over a region available to flow that is large with respect to the pore size. Equation 4C.3-2 was proposed empirically to describe the slow seepage of fluids through granular media.

When Eqs. 4C.3-1 and 2 are combined we get

$$\left(\frac{\varepsilon \mu}{\kappa}\right) \frac{\partial \rho}{\partial t} = (\nabla \cdot \rho(\nabla p - \rho \mathbf{g})) \tag{4C.3-3}$$

for constant viscosity and permeability. This equation and the equation of state describe the motion of a fluid in a porous medium. For most purposes we may write the *equation of state* as

$$\rho = \rho_0 p^m e^{\beta p} \tag{4C.3-4}$$

in which ρ_0 is the fluid density at unit pressure, and the following parameters have been given:[6]

1. Incompressible liquids	$m = 0$	$\beta = 0$
2. Compressible liquids	$m = 0$	$\beta \neq 0$
3. Isothermal expansion of gases	$\beta = 0$	$m = 1$
4. Adiabatic expansion of gases	$\beta = 0$	$m = C_V/C_p = 1/\gamma$

Show that Eqs. 4C.3-3 and 4 can be combined and simplified for these four categories to give (for gases it is customary to neglect the gravity terms since they are small compared with the pressure terms):

Case 1. $$\nabla^2 \mathscr{P} = 0 \tag{4C.3-5}$$

Case 2. $$\left(\frac{\varepsilon \mu \beta}{\kappa}\right) \frac{\partial \rho}{\partial t} = \nabla^2 \rho - (\nabla \cdot \rho^2 \beta \mathbf{g}) \tag{4C.3-6}$$

Case 3. $$\left(\frac{2\varepsilon \mu \rho_0}{\kappa}\right) \frac{\partial \rho}{\partial t} = \nabla^2 \rho^2 \tag{4C.3-7}$$

Case 4. $$\left(\frac{(m+1)\varepsilon \mu \rho_0^{1/m}}{\kappa}\right) \frac{\partial \rho}{\partial t} = \nabla^2 \rho^{(1+m)/m} \tag{4C.3-8}$$

[5]**Henry Philibert Gaspard Darcy** (1803–1858) studied in Paris and became famous for designing the municipal water-supply system in Dijon, the city of his birth. H. Darcy, *Les Fontaines Publiques de la Ville de Dijon*, Victor Dalmont, Paris (1856). For further discussion of "Darcy's law," see J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics*, Martinus Nihjoff, Dordrecht (1983); and H. Brenner and D. A. Edwards, *Macrotransport Processes*, Butterworth-Heinemann, Boston (1993).

Note that Case 1 leads to *Laplace's equation*, Case 2 without the gravity term leads to the *heat conduction* or *diffusion equation*, and Cases 3 and 4 lead to nonlinear equations.[7]

[6]M. Muskat, *Flow of Homogeneous Fluids Through Porous Media*, McGraw-Hill (1937).

[7]For the boundary condition at a porous surface that bounds a moving fluid, see G. S. Beavers and D. D. Joseph, *J. Fluid Mech.*, **30**, 197-207 (1967) and G. S. Beavers, E. M. Sparrow, and B. A. Masha, *AIChE Journal*, **20**, 596-597 (1974).