# Problem 4D.2

**Start-up of laminar flow in a circular tube** (Fig. 4D.2). A fluid of constant density and viscosity is contained in a very long pipe of length $L$ and radius $R$. Initially the fluid is at rest. At time $t = 0$, a pressure gradient $(\mathscr{P}_0 - \mathscr{P}_L)/L$ is imposed on the system. Determine how the velocity profiles change with time.

**(a)** Show that the relevant equation of motion can be put into dimensionless form as follows:

$$\frac{\partial \phi}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi}\right) \tag{4D.2-1}$$

in which $\xi = r/R$, $\tau = \mu t/\rho R^2$, and $\phi = [(\mathscr{P}_0 - \mathscr{P}_L)R^2/4\mu L]^{-1} v_z$.

**(b)** Show that the asymptotic solution for large time is $\phi_\infty = 1 - \xi^2$. Then define $\phi_t$ by $\phi(\xi, \tau) = \phi_\infty(\xi) - \phi_t(\xi, \tau)$, and solve the partial differential equation for $\phi_t$ by the method of separation of variables.

**(c)** Show that the final solution is

$$\phi(\xi, \tau) = (1 - \xi^2) - 8\sum_{n=1}^{\infty} \frac{J_0(\alpha_n \xi)}{\alpha_n^3 J_1(\alpha_n)} \exp(-\alpha_n^2 \tau) \tag{4D.2-2}$$

in which $J_n(\xi)$ is the $n$th order Bessel function of $\xi$, and the $\alpha_n$ are the roots of the equation $J_0(\alpha_n) = 0$. The result is plotted in Fig. 4D.2.

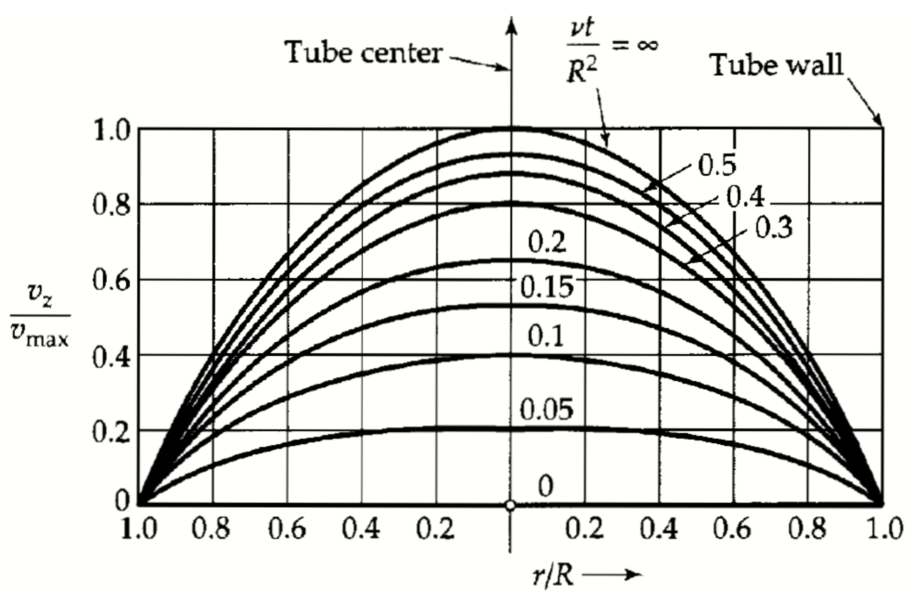


**Fig. 4D.2.** Velocity distribution for the unsteady flow resulting from a suddenly impressed pressure gradient in a circular tube [P. Szymanski, *J. Math. Pures Appl.*, Series 9, **11**, 67–107 (1932)].