Problem 4D.3

Flows in the disk-and-tube system (Fig. 4D.3).[9]

(a) A fluid in a circular tube is caused to move tangentially by a tightly fitting rotating disk at the liquid surface at $z = 0$; the bottom of the tube is located at $z = L$. Find the steady-state velocity distribution $v_\theta(r, z)$, when the angular velocity of the disk is Ω. Assume that creeping flow prevails throughout, so that there is no secondary flow. Find the limit of the solution as $L \to \infty$.

(b) Repeat the problem for the unsteady flow. The fluid is at rest before $t = 0$, and the disk suddenly begins to rotate with an angular velocity Ω at $t = 0$. Find the velocity distribution $v_\theta(r, z, t)$ for a column of fluid of height L. Then find the solution for the limit as $L \to \infty$.

(c) If the disk is oscillating sinusoidally in the tangential direction with amplitude Ω_0, obtain the velocity distribution in the tube when the "oscillatory steady state" has been attained. Repeat the problem for a tube of infinite length.

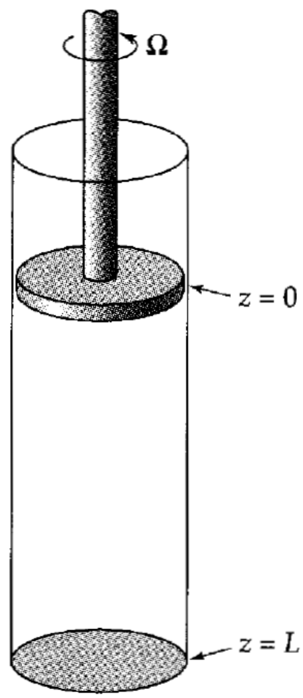

Fig. 4D.3. Rotating disk in a circular tube.

Solution

[9]W. Hort, *Z. tech. Phys.*, **10**, 213 (1920); C. T. Hill, J. D. Huppler, and R. B. Bird, *Chem. Engr. Sci.*, **21**, 815–817 (1966).

Part (a)

Since the flow is tangential, we assume that the velocity varies as a function of radius and height and that the fluid moves only in the θ-direction.

$$\mathbf{v} = v_\theta(r, z)\hat{\boldsymbol{\theta}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $r = a$ (the tube radius), $z = 0$, and $z = L$. Note that the angular velocity vector points in the direction of $-z$ $\boldsymbol{\Omega} = -\Omega\hat{\mathbf{z}}$, so the tangential velocity on the disk is $-\Omega r$.

$$\begin{aligned} &\text{Boundary Condition 1:} \qquad v_\theta(a, z) = 0 \\ &\text{Boundary Condition 2:} \qquad v_\theta(r, 0) = -\Omega r \\ &\text{Boundary Condition 3:} \qquad v_\theta(r, L) = 0 \end{aligned}$$

Because the velocity is zero at the disk center and at the bottom, we further assume that the velocity is zero all along the tube axis.

$$\text{Boundary Condition 4:} \qquad v_\theta(0, z) = 0$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

The fact that creeping flow prevails throughout means the acceleration terms on the left side are equal to zero.

$$\mathbf{0} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using cylindrical coordinates is the appropriate choice for this problem, so equations (1) and (2) will be used in (r, θ, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=\,0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=\,0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=\,0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$0 = -\frac{\partial p}{\partial r} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=\,0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=\,0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=\,0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=\,0}\right] + \underbrace{\rho g_r}_{=\,0}$$

$$0 = \underbrace{-\frac{1}{r}\frac{\partial p}{\partial \theta}}_{=\,0} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=\,0} + \frac{\partial^2 v_\theta}{\partial z^2} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=\,0}\right] + \underbrace{\rho g_\theta}_{=\,0}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=\,0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=\,0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=\,0}\right] + \rho g_z$$

The relevant equation for the velocity is the θ-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_\theta(r,z)\hat{\boldsymbol{\theta}}$.

$$0 = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{\partial^2 v_\theta}{\partial z^2}$$

Because this differential equation and the boundary conditions associated with it are linear and homogeneous (except for one), the method of separation of variables can be applied to solve the equation. Assume a product solution of the form $v_\theta = R(r)Z(z)$ and plug it into the PDE

$$0 = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}[rR(r)Z(z)]\right) + \frac{\partial^2 [R(r)Z(z)]}{\partial z^2} \quad \rightarrow \quad 0 = Z\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rR)\right] + RZ''$$

and the boundary conditions.

$$\begin{aligned} v_\theta(r,L) = 0 \quad &\rightarrow \quad R(r)Z(L) = 0 \quad \rightarrow \quad Z(L) = 0 \\ v_\theta(0,z) = 0 \quad &\rightarrow \quad R(0)Z(z) = 0 \quad \rightarrow \quad R(0) = 0 \\ v_\theta(a,z) = 0 \quad &\rightarrow \quad R(a)Z(z) = 0 \quad \rightarrow \quad R(a) = 0 \end{aligned}$$

Now separate variables in the PDE: bring all functions of z to the left side and all functions of r to the right side. Note that the final answer will be the same regardless of which side the minus sign is on.

$$-\frac{Z''}{Z} = \frac{1}{R}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rR)\right]$$

The only way a function of z can be equal to a function of r is if both sides are equal to a constant λ.

$$-\frac{Z''}{Z} = \frac{1}{R}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rR)\right] = \lambda$$

Values of λ for which $R(0) = 0$ and $R(a) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $R(r)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \eta^2$

Assuming λ is positive, the differential equation for R becomes

$$\frac{1}{R}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rR)\right] = \eta^2.$$

Multiply both sides by $r^2 R$.

$$r^2\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rR)\right] = \eta^2 r^2 R$$

Expand the left side.

$$r^2R'' + rR' - R = \eta^2 r^2 R$$

Bring all terms to the left side and factor R.

$$r^2R'' + rR' + [(i\eta)^2 r^2 - 1]R = 0$$

This is the parametric form of Bessel's equation of order 1. Its solution can be written in terms of first-order modified Bessel functions of the first and second kind, $I_1(\eta r)$ and $K_1(\eta r)$, respectively.

$$R(r) = C_1 I_1(\eta r) + C_2 K_1(\eta r)$$

In order to satisfy $R(0) = 0$ and $R(a) = 0$, both C_1 and C_2 must be set to zero because I_1 and K_1 are not oscillatory. Only the trivial solution $R(r) = 0$ is obtained, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Assuming λ is zero, the differential equation for R becomes

$$\frac{1}{R}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rR)\right] = 0.$$

Multiply both sides by R.

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rR)\right] = 0$$

Integrate both sides with respect to r.

$$\frac{1}{r}\frac{d}{dr}(rR) = C_3$$

Multiply both sides by r.

$$\frac{d}{dr}(rR) = C_3 r$$

Integrate both sides with respect to r once more.

$$rR = C_3\frac{r^2}{2} + C_4$$

Divide both sides by r.

$$R(r) = C_3\frac{r}{2} + \frac{C_4}{r}$$

For $R(0) = 0$ to be satisfied, we require $C_4 = 0$. Apply the other boundary condition at $r = a$ to determine C_3.

$$R(a) = C_3\frac{a}{2} = 0 \quad \rightarrow \quad C_3 = 0$$

As a result, only the trivial solution $R(r) = 0$ is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Assuming λ is negative, the differential equation for R becomes

$$\frac{1}{R}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rR)\right] = -\gamma^2.$$

Multiply both sides by r^2R.

$$r^2\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rR)\right] = -\gamma^2r^2R$$

Expand the left side.

$$r^2R'' + rR' - R = -\gamma^2r^2R$$

Bring all terms to the left side and factor R.

$$r^2R'' + rR' + (\gamma^2r^2 - 1)R = 0$$

This is the parametric form of Bessel's equation of order 1. Its solution can be written in terms of first-order Bessel functions of the first and second kind, $J_1(\gamma r)$ and $Y_1(\gamma r)$, respectively.

$$R(r) = C_5 J_1(\gamma r) + C_6 Y_1(\gamma r)$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} R(0) &= C_5 J_1(0) + C_6 Y_1(0) = 0 \\ R(a) &= C_5 J_1(\gamma a) + C_6 Y_1(\gamma a) = 0 \end{aligned}$$

Note that $J_1(0) = 0$ and $Y_1(0) = -\infty$, so the first equation is satisfied by setting $C_6 = 0$. That makes the second equation $C_5 J_1(\gamma a) = 0$. In order to avoid getting the trivial solution, we insist that $C_5 \neq 0$. Then we have

$$J_1(\gamma a) = 0.$$

We conclude that $\gamma a = \alpha_{1n}$ $(n = 1, 2, \ldots)$, where α_{1n} is the nth zero of J_1. The eigenfunctions corresponding to these eigenvalues are

$$R(r) = C_5 J_1(\gamma r) \quad \rightarrow \quad R_n(r) = J_1\left(\frac{\alpha_{1n}}{a} r\right), \quad n = 1, 2, \ldots.$$

Now the ODE for Z will be solved.

$$-\frac{Z''}{Z} = -\gamma^2$$

Multiply both sides by $-Z$.

$$Z'' = \gamma^2 Z$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Z(z) = C_7 \cosh \gamma z + C_8 \sinh \gamma z$$

Apply the boundary condition at $z = L$ to determine C_7.

$$Z(L) = C_7 \cosh \gamma L + C_8 \sinh \gamma L = 0 \quad \rightarrow \quad C_7 = -C_8 \frac{\sinh \gamma L}{\cosh \gamma L}$$

So we have

$$\begin{aligned} Z(z) &= -C_8 \frac{\sinh \gamma L}{\cosh \gamma L} \cosh \gamma z + C_8 \sinh \gamma z \\ &= -C_8 \frac{\sinh \gamma L \cosh \gamma z - \sinh \gamma z \cosh \gamma L}{\cosh \gamma L} \\ &= -C_8 \frac{\sinh[\gamma(L - z)]}{\cosh \gamma L} \\ &= C_9 \sinh[\gamma(L - z)], \end{aligned}$$

and the eigenfunctions are

$$Z_n(z) = \sinh\left[\frac{\alpha_{1n}}{a}(L - z)\right], \quad n = 1, 2, \ldots.$$

According to the principle of superposition, the solution to the PDE for $v_\theta(r, z)$ is a linear combination of all products $R_n(r) Z_n(z)$ over all the eigenvalues.

$$v_\theta(r, z) = \sum_{n=1}^{\infty} A_n J_1\left(\frac{\alpha_{1n}}{a} r\right) \sinh\left[\frac{\alpha_{1n}}{a}(L - z)\right]$$

The aim now is to use the inhomogeneous boundary condition at $z = 0$ to determine the coefficients A_n.

$$v_\theta(r, 0) = \sum_{n=1}^{\infty} A_n J_1\left(\frac{\alpha_{1n}}{a} r\right) \sinh\left(\frac{\alpha_{1n}}{a} L\right) = -\Omega r$$

Multiply both sides by $r J_1\left(\frac{\alpha_{1m}}{a} r\right)$, where m is an integer.

$$\sum_{n=1}^{\infty} A_n \sinh\left(\frac{\alpha_{1n}}{a} L\right) J_1\left(\frac{\alpha_{1n}}{a} r\right) J_1\left(\frac{\alpha_{1m}}{a} r\right) r = -\Omega r^2 J_1\left(\frac{\alpha_{1m}}{a} r\right)$$

Integrate both sides with respect to r from 0 to a.

$$\int_0^a \sum_{n=1}^{\infty} A_n \sinh\left(\frac{\alpha_{1n}}{a} L\right) J_1\left(\frac{\alpha_{1n}}{a} r\right) J_1\left(\frac{\alpha_{1m}}{a} r\right) r\, dr = -\int_0^a \Omega r^2 J_1\left(\frac{\alpha_{1m}}{a} r\right) dr$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} A_n \sinh\left(\frac{\alpha_{1n}}{a} L\right) \int_0^a J_1\left(\frac{\alpha_{1n}}{a} r\right) J_1\left(\frac{\alpha_{1m}}{a} r\right) r\, dr = -\Omega \int_0^a r^2 J_1\left(\frac{\alpha_{1m}}{a} r\right) dr$$

Because the Bessel functions are orthogonal with one another on $[0, a]$ with respect to the weight r, the integral on the left side is zero for $n \neq m$. The $n = m$ term is all that remains in the infinite series.

$$A_n \sinh\left(\frac{\alpha_{1n}}{a} L\right) \int_0^a J_1^2\left(\frac{\alpha_{1n}}{a} r\right) r\, dr = -\Omega \int_0^a r^2 J_1\left(\frac{\alpha_{1n}}{a} r\right) dr$$

Both of these integrals are known.

$$A_n \sinh\left(\frac{\alpha_{1n}}{a} L\right) \cdot \frac{a^2}{2} J_2^2(\alpha_{1n}) = -\Omega \cdot \frac{a^3}{\alpha_{1n}} J_2(\alpha_{1n}),$$

where J_2 is the second-order Bessel function of the first kind. Solve the equation for A_n.

$$A_n = -\frac{2\Omega a}{\alpha_{1n} \sinh\left(\frac{\alpha_{1n}}{a} L\right) J_2(\alpha_{1n})}$$

Here we apply a known property of Bessel functions, $J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$, to write J_2 in terms of J_1 and J_0.

$$A_n = -\frac{2\Omega a}{\alpha_{1n} \sinh\left(\frac{\alpha_{1n}}{a} L\right) \left[\frac{2}{\alpha_{1n}} \underbrace{J_1(\alpha_{1n})}_{=\,0} - J_0(\alpha_{1n})\right]} = \frac{2\Omega a}{\alpha_{1n} \sinh\left(\frac{\alpha_{1n}}{a} L\right) J_0(\alpha_{1n})}$$

We have for v_θ then

$$v_\theta(r, z) = \sum_{n=1}^{\infty} \frac{2\Omega a}{\alpha_{1n} \sinh\left(\frac{\alpha_{1n}}{a} L\right) J_0(\alpha_{1n})} J_1\left(\frac{\alpha_{1n}}{a} r\right) \sinh\left[\frac{\alpha_{1n}}{a}(L - z)\right].$$

Therefore,

$$\boxed{v_\theta(r, z) = \sum_{n=1}^{\infty} \frac{2\Omega a}{\alpha_{1n}} \frac{J_1\left(\frac{\alpha_{1n}}{a} r\right)}{J_0(\alpha_{1n})} \frac{\sinh\left[\frac{\alpha_{1n}}{a}(L - z)\right]}{\sinh\left(\frac{\alpha_{1n}}{a} L\right)}}.$$

The velocity field can be plotted in Cartesian coordinates by replacing r with $\sqrt{x^2+y^2}$ and $\hat{\boldsymbol{\theta}}$ with $-\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}}$.

$$\mathbf{v} = v_\theta(r,z)\hat{\boldsymbol{\theta}} = v_\theta(\sqrt{x^2+y^2},z)(-\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}}) = v_\theta(\sqrt{x^2+y^2},z)\left(-\frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{x}}+\frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{y}}\right)$$

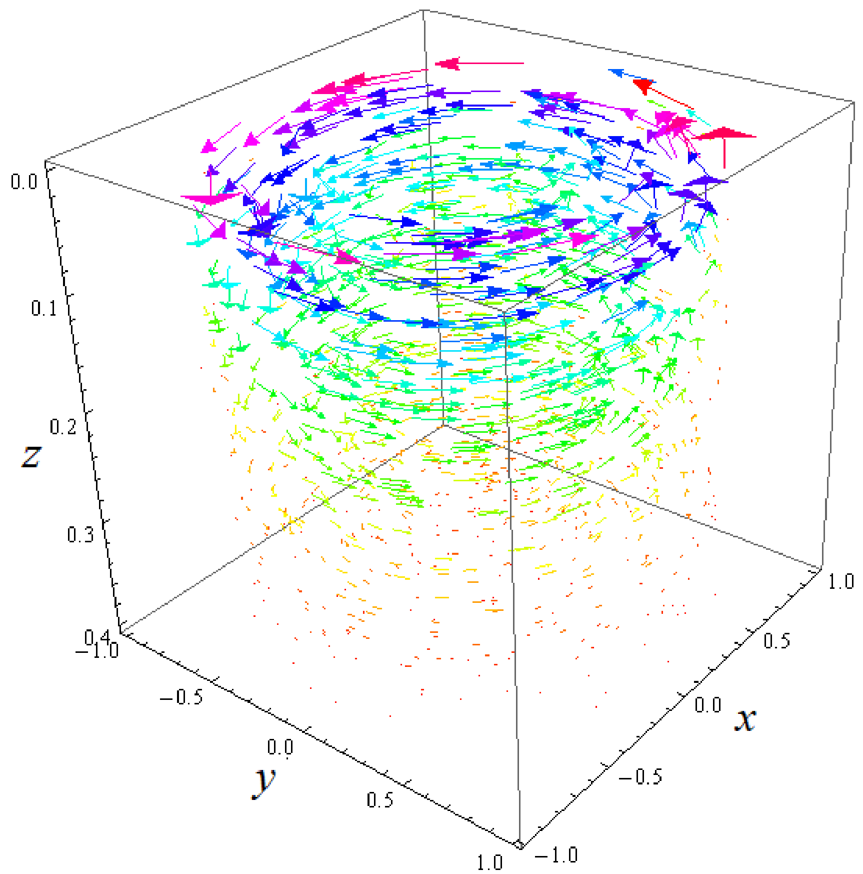

Figure 1: This figure shows the steady-state velocity field $\mathbf{v}$ for $\Omega = 1$, $a = 1$, and $L = 0.4$. It is approximate, as only the first 50 terms in the infinite series have been used.

To find the limit of v_θ as $L\to\infty$, note that

$$\begin{aligned}\lim_{L\to\infty}\frac{\sinh[\gamma(L-z)]}{\sinh\gamma L} = \lim_{L\to\infty}\frac{\sinh\gamma L\cosh\gamma z-\cosh\gamma L\sinh\gamma z}{\sinh\gamma L} &= \cosh\gamma z-\sinh\gamma z\underbrace{\lim_{L\to\infty}\coth\gamma L}_{=1}\\ &= \cosh\gamma z-\sinh\gamma z\\ &= e^{-\gamma z}.\end{aligned}$$

Therefore,

$$\boxed{\lim_{L\to\infty}v_\theta(r,z) = \sum_{n=1}^{\infty}\frac{2\Omega a}{\alpha_{1n}}\frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})}\exp\left(-\frac{\alpha_{1n}}{a}z\right).}$$

Part (b)

Since the flow is unsteady and tangential, we assume that the velocity varies as a function of radius, height, and time and that the fluid moves only in the θ-direction.

$$\mathbf{v} = v_\theta(r, z, t)\hat{\boldsymbol{\theta}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $r = a$ (the tube radius), $z = 0$, and $z = L$. Note that the angular velocity vector points in the direction of $-z$ $\boldsymbol{\Omega} = -\Omega\hat{\mathbf{z}}$, so the tangential velocity on the disk is $-\Omega r$.

$$\begin{aligned} \text{Boundary Condition 1:} &\qquad v_\theta(a, z, t) = 0 \\ \text{Boundary Condition 2:} &\qquad v_\theta(r, 0, t) = -\Omega r \\ \text{Boundary Condition 3:} &\qquad v_\theta(r, L, t) = 0 \end{aligned}$$

Because the velocity is zero at the disk center and at the bottom, we further assume that the velocity is zero all along the tube axis.

$$\text{Boundary Condition 4:} \qquad v_\theta(0, z, t) = 0$$

Also, the fluid starts from rest, so the velocity is zero initially.

$$\text{Initial Condition:} \qquad v_\theta(r, z, 0) = 0$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using cylindrical coordinates is the appropriate choice for this problem, so the two previous equations will be used in (r, θ, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\Bigg(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \frac{v_\theta^2}{r}\Bigg) = -\frac{\partial p}{\partial r} + \mu\Bigg[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\Bigg] + \underbrace{\rho g_r}_{=0}$$

$$\rho\Bigg(\frac{\partial v_\theta}{\partial t} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\Bigg) = -\underbrace{\frac{1}{r}\frac{\partial p}{\partial \theta}}_{=0} + \mu\Bigg[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \frac{\partial^2 v_\theta}{\partial z^2} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\Bigg] + \underbrace{\rho g_\theta}_{=0}$$

$$\rho\Bigg(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\Bigg) = -\frac{\partial p}{\partial z} + \mu\Bigg[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\Bigg] + \rho g_z$$

The relevant equation for the velocity is the θ-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_\theta(r,z,t)\hat{\boldsymbol{\theta}}$.

$$\rho\frac{\partial v_\theta}{\partial t} = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

As a result of the spinning disk at $z = 0$, the fluid starts to move and eventually reaches the steady-state solution calculated in part (a). Let the boxed solution for $v_\theta(r,z)$ in part (a) be denoted here as $v_\infty(r,z)$. The velocity $v_\theta(r,z,t)$ can be thought to have an equilibrium part $v_\infty(r,z)$, independent of time, and a transient part $v_t(r,z,t)$: $v_\theta(r,z,t) = v_\infty(r,z) - v_t(r,z,t)$.

$$\rho\frac{\partial}{\partial t}[v_\infty(r,z) - v_t(r,z,t)] = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r[v_\infty(r,z) - v_t(r,z,t)])\right) + \frac{\partial^2[v_\infty(r,z) - v_t(r,z,t)]}{\partial z^2}\right]$$

$$-\rho\frac{\partial v_t}{\partial t} = \underbrace{\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\infty)\right) + \frac{\partial^2 v_\infty}{\partial z^2}\right]}_{=0} - \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_t)\right) + \frac{\partial^2 v_t}{\partial z^2}\right]$$

Multiply both sides by -1 to get the PDE for v_t.

$$\rho\frac{\partial v_t}{\partial t} = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_t)\right) + \frac{\partial^2 v_t}{\partial z^2}\right] \tag{3}$$

Now we will obtain the initial and boundary conditions for v_t.

$$\begin{aligned}
v_\theta(0,z,t) = v_\infty(0,z) - v_t(0,z,t) = 0 \quad &\rightarrow \quad 0 - v_t(0,z,t) = 0 \quad &\rightarrow \quad v_t(0,z,t) = 0 \\
v_\theta(a,z,t) = v_\infty(a,z) - v_t(a,z,t) = 0 \quad &\rightarrow \quad 0 - v_t(a,z,t) = 0 \quad &\rightarrow \quad v_t(a,z,t) = 0 \\
v_\theta(r,0,t) = v_\infty(r,0) - v_t(r,0,t) = -\Omega r \quad &\rightarrow \quad -\Omega r - v_t(r,0,t) = -\Omega r \quad &\rightarrow \quad v_t(r,0,t) = 0 \\
v_\theta(r,L,t) = v_\infty(r,L) - v_t(r,L,t) = 0 \quad &\rightarrow \quad 0 - v_t(r,L,t) = 0 \quad &\rightarrow \quad v_t(r,L,t) = 0 \\
v_\theta(r,z,0) = v_\infty(r,z) - v_t(r,z,0) = 0 \quad & \quad &\rightarrow \quad v_t(r,z,0) = v_\infty(r,z)
\end{aligned}$$

The reason we're solving for v_t instead of v_θ directly is that the boundary conditions for v_t are all homogeneous. This allows us to use the method of separation of variables to solve the PDE with the given initial condition. Assume a product solution of the form $v_t = R(r)Z(z)T(t)$ and plug it into the PDE

$$\rho\frac{\partial[R(r)Z(z)T(t)]}{\partial t} = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}[rR(r)Z(z)T(t)]\right) + \frac{\partial^2[R(r)Z(z)T(t)]}{\partial z^2}\right]$$

$$\rightarrow \quad \rho RZT' = \mu\left[ZT\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rR)\right) + RTZ''\right]$$

and the boundary conditions.

$$\begin{aligned}
v_t(0,z,t) = 0 \quad &\rightarrow \quad R(0)Z(z)T(t) = 0 \quad &\rightarrow \quad R(0) = 0 \\
v_t(a,z,t) = 0 \quad &\rightarrow \quad R(a)Z(z)T(t) = 0 \quad &\rightarrow \quad R(a) = 0 \\
v_t(r,0,t) = 0 \quad &\rightarrow \quad R(r)Z(0)T(t) = 0 \quad &\rightarrow \quad Z(0) = 0 \\
v_t(r,L,t) = 0 \quad &\rightarrow \quad R(r)Z(L)T(t) = 0 \quad &\rightarrow \quad Z(L) = 0
\end{aligned}$$

Now separate variables in the PDE: divide both sides by μRZT to bring all functions of t to the left side and all functions of r and z to the right side. The final answer will be the same regardless of which side the constants are on.

$$\frac{\rho}{\mu}\frac{T'}{T} = \frac{1}{R}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rR)\right) + \frac{Z''}{Z}$$

The only way a function of t can be equal to a function of r and z is if both sides are equal to a constant β.

$$\frac{\rho}{\mu}\frac{T'}{T} = \frac{1}{R}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rR)\right) + \frac{Z''}{Z} = \beta$$

In addition, if we bring Z''/Z to the right side, then we find that a function of r is equal to a function of z, another constant λ.

$$\frac{1}{R}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rR)\right) = \beta - \frac{Z''}{Z} = \lambda$$

In summary, using the method of separation of variables reduces the PDE for v_t in equation (3) to three ODEs, one for each variable.

$$\begin{aligned} \frac{\rho}{\mu}\frac{T'}{T} &= \beta \\ \frac{1}{R}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rR)\right) &= \lambda \\ \beta - \frac{Z''}{Z} &= \lambda \end{aligned}$$

Values of β and λ for which $R(0) = 0$, $R(a) = 0$, $Z(0) = 0$, and $Z(L) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions, $R(r)$ and $Z(z)$, associated with them are called the eigenfunctions. The equation for $R(r)$ was already solved in part (a). There we found that $\lambda = -\gamma^2$, where $\gamma a = \alpha_{1n}$, and

$$R(r) = C_5 J_1(\gamma r) \quad \rightarrow \quad R_n(r) = J_1\left(\frac{\alpha_{1n}}{a}r\right), \quad n = 1, 2, \ldots.$$

Consequently, the equation for Z becomes

$$\beta - \frac{Z''}{Z} = -\gamma^2.$$

Determination of Positive Eigenvalues: $\beta = \xi^2$

Assuming β is positive, we get

$$\xi^2 - \frac{Z''}{Z} = -\gamma^2.$$

Bring ξ^2 to the right and multiply both sides by $-Z$.

$$Z'' = (\gamma^2 + \xi^2)Z$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$Z(z) = C_{10}\cosh(\sqrt{\gamma^2 + \xi^2}z) + C_{11}\sinh(\sqrt{\gamma^2 + \xi^2}z)$$

Apply the boundary conditions here to determine C_{10} and C_{11}.

$$\begin{aligned} &Z(0) = C_{10} = 0 \\ &Z(L) = C_{10}\cosh(\sqrt{\gamma^2 + \xi^2}L) + C_{11}\sinh(\sqrt{\gamma^2 + \xi^2}L) = 0 \end{aligned}$$

For the second equation to be satisfied, C_{11} must be equal to zero because sinh is not oscillatory. Only the trivial solution is obtained, $Z(z) = 0$, so there are no positive eigenvalues for β.

Determination of the Zero Eigenvalue: $\beta = 0$

Assuming β is zero, we get

$$-\frac{Z''}{Z} = -\gamma^2.$$

Multiply both sides by $-Z$.

$$Z'' = \gamma^2 Z$$

Again, the general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$Z(z) = C_{12} \cosh \gamma z + C_{13} \sinh \gamma z$$

Apply the boundary conditions here to determine C_{12} and C_{13}.

$$\begin{aligned} Z(0) &= C_{12} = 0 \\ Z(L) &= C_{12} \cosh \gamma L + C_{13} \sinh \gamma L = 0 \end{aligned}$$

For the second equation to be satisfied, C_{13} must be equal to zero because sinh is not oscillatory. Only the trivial solution is obtained, $Z(z) = 0$, so zero is not an eigenvalue for β.

Determination of Negative Eigenvalues: $\beta = -\zeta^2$

Assuming β is negative, we get

$$-\zeta^2 - \frac{Z''}{Z} = -\gamma^2.$$

Bring ζ^2 to the right and multiply both sides by $-Z$.

$$Z'' = -(\zeta^2 - \gamma^2) Z$$

The general solution is written in terms of sine and cosine.

$$Z(z) = C_{14} \cos(\sqrt{\zeta^2 - \gamma^2} z) + C_{15} \sin(\sqrt{\zeta^2 - \gamma^2} z)$$

Apply the boundary conditions here to determine C_{14} and C_{15}.

$$\begin{aligned} Z(0) &= C_{14} = 0 \\ Z(L) &= C_{14} \cos(\sqrt{\zeta^2 - \gamma^2} L) + C_{15} \sin(\sqrt{\zeta^2 - \gamma^2} L) = 0 \end{aligned}$$

The second equation simplifies to $C_{15} \sin(\sqrt{\zeta^2 - \gamma^2} L) = 0$. In order to avoid getting the trivial solution, we insist that $C_{15} \neq 0$. Then we have

$$\sin(\sqrt{\zeta^2 - \gamma^2} L) = 0.$$

The sine's argument must then be a positive integer multiple of π.

$$\sqrt{\zeta^2 - \gamma^2} L = m\pi, \quad m = 1, 2, \ldots$$

Solve for ζ now.

$$\sqrt{\zeta^2-\gamma^2}=\frac{m\pi}{L} \quad\rightarrow\quad \zeta^2=\gamma^2+\frac{m^2\pi^2}{L^2} \quad\rightarrow\quad \zeta_{mn}=\sqrt{\frac{\alpha_{1n}^2}{a^2}+\frac{m^2\pi^2}{L^2}}, \quad \begin{array}{r} m=1,2,\ldots \\ n=1,2,\ldots \end{array}$$

The eigenfunctions associated with these eigenvalues are

$$Z(z)=C_{15}\sin(\sqrt{\zeta^2-\gamma^2}z) \quad\rightarrow\quad Z_m(z)=\sin\frac{m\pi z}{L}, \quad m=1,2,\ldots.$$

As a result, the ODE for T becomes

$$\frac{\rho}{\mu}\frac{T'}{T}=-\zeta^2.$$

Multiply both sides by $\mu T/\rho$.

$$T'=-\frac{\mu}{\rho}\zeta^2 T$$

The general solution is in terms of the exponential function.

$$T(t)=C_{16}\exp\left(-\frac{\mu}{\rho}\zeta^2 t\right) \quad\rightarrow\quad T_{mn}(t)=\exp\left[-\frac{\mu}{\rho}\left(\frac{\alpha_{1n}^2}{a^2}+\frac{m^2\pi^2}{L^2}\right)t\right]$$

According to the principle of superposition, the solution to the PDE for $v_t(r,z,t)$ is a linear combination of all products $R_n(r)Z_m(z)T_{mn}(t)$ over all the eigenvalues.

$$v_t(r,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\exp\left[-\frac{\mu}{\rho}\left(\frac{\alpha_{1n}^2}{a^2}+\frac{m^2\pi^2}{L^2}\right)t\right]J_1\left(\frac{\alpha_{1n}}{a}r\right)\sin\frac{m\pi z}{L}$$

Our task now is to use the initial condition to determine B_{mn}.

$$v_t(r,z,0)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}J_1\left(\frac{\alpha_{1n}}{a}r\right)\sin\frac{m\pi z}{L}=v_\infty(r,z),$$

or

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}J_1\left(\frac{\alpha_{1n}}{a}r\right)\sin\frac{m\pi z}{L}=\sum_{n=1}^{\infty}\frac{2\Omega a}{\alpha_{1n}}\frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})}\frac{\sinh\left[\frac{\alpha_{1n}}{a}(L-z)\right]}{\sinh\left(\frac{\alpha_{1n}}{a}L\right)}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}B_{mn}\sin\frac{m\pi z}{L}\right)J_1\left(\frac{\alpha_{1n}}{a}r\right)=\sum_{n=1}^{\infty}\left\{\frac{2\Omega a}{\alpha_{1n}}\frac{1}{J_0(\alpha_{1n})}\frac{\sinh\left[\frac{\alpha_{1n}}{a}(L-z)\right]}{\sinh\left(\frac{\alpha_{1n}}{a}L\right)}\right\}J_1\left(\frac{\alpha_{1n}}{a}r\right).$$

Thus,

$$\sum_{m=1}^{\infty}B_{mn}\sin\frac{m\pi z}{L}=\frac{2\Omega a}{\alpha_{1n}}\frac{1}{J_0(\alpha_{1n})}\frac{\sinh\left[\frac{\alpha_{1n}}{a}(L-z)\right]}{\sinh\left(\frac{\alpha_{1n}}{a}L\right)}.$$

To solve for B_{mn}, multiply both sides by $\sin(p\pi z/L)$, where p is an integer.

$$\sum_{m=1}^{\infty}B_{mn}\sin\frac{m\pi z}{L}\sin\frac{p\pi z}{L}=\frac{2\Omega a}{\alpha_{1n}}\frac{1}{J_0(\alpha_{1n})}\frac{\sinh\left[\frac{\alpha_{1n}}{a}(L-z)\right]}{\sinh\left(\frac{\alpha_{1n}}{a}L\right)}\sin\frac{p\pi z}{L}$$

Integrate both sides with respect to z from 0 to L.

$$\int_0^L \sum_{m=1}^{\infty} B_{mn} \sin \frac{m\pi z}{L} \sin \frac{p\pi z}{L}\, dz = \int_0^L \frac{2\Omega a}{\alpha_{1n}} \frac{1}{J_0(\alpha_{1n})} \frac{\sinh\left[\frac{\alpha_{1n}}{a}(L-z)\right]}{\sinh\left(\frac{\alpha_{1n}}{a}L\right)} \sin \frac{p\pi z}{L}\, dz$$

Bring the constants in front of the integrals.

$$\sum_{m=1}^{\infty} B_{mn} \int_0^L \sin \frac{m\pi z}{L} \sin \frac{p\pi z}{L}\, dz = \frac{2\Omega a}{\alpha_{1n}} \frac{1}{J_0(\alpha_{1n})} \frac{1}{\sinh\left(\frac{\alpha_{1n}}{a}L\right)} \int_0^L \sinh\left[\frac{\alpha_{1n}}{a}(L-z)\right] \sin \frac{p\pi z}{L}\, dz$$

Because the sine functions are orthogonal on the interval $[0, L]$, the integral on the left side is zero for $m \neq p$. The $m = p$ term is all that remains in the infinite series.

$$B_{mn} \int_0^L \sin^2 \frac{m\pi z}{L}\, dz = \frac{2\Omega a}{\alpha_{1n}} \frac{1}{J_0(\alpha_{1n})} \frac{1}{\sinh\left(\frac{\alpha_{1n}}{a}L\right)} \int_0^L \sinh\left[\frac{\alpha_{1n}}{a}(L-z)\right] \sin \frac{m\pi z}{L}\, dz$$

Evaluate the integrals.

$$B_{mn} \cdot \frac{L}{2} = \frac{2\Omega a}{\alpha_{1n}} \frac{1}{J_0(\alpha_{1n})} \frac{1}{\sinh\left(\frac{\alpha_{1n}}{a}L\right)} \cdot \frac{\frac{m\pi}{L}}{\frac{\alpha_{1n}^2}{a^2} + \frac{m^2\pi^2}{L^2}} \sinh\left(\frac{\alpha_{1n}}{a}L\right)$$

Solve the equation for B_{mn}.

$$B_{mn} = \frac{2}{L} \frac{2\Omega a}{\alpha_{1n}} \frac{\frac{m\pi}{L}}{\frac{\alpha_{1n}^2}{a^2} + \frac{m^2\pi^2}{L^2}} \frac{1}{J_0(\alpha_{1n})}$$

The solution for v_t is now known.

$$\begin{aligned} v_t(r, z, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{2}{L} \frac{2\Omega a}{\alpha_{1n}} \frac{\frac{m\pi}{L}}{\frac{\alpha_{1n}^2}{a^2} + \frac{m^2\pi^2}{L^2}} \frac{1}{J_0(\alpha_{1n})} \exp\left[-\frac{\mu}{\rho}\left(\frac{\alpha_{1n}^2}{a^2} + \frac{m^2\pi^2}{L^2}\right)t\right] J_1\left(\frac{\alpha_{1n}}{a}r\right) \sin \frac{m\pi z}{L} \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{2}{L} \frac{2\Omega a}{\alpha_{1n}} \frac{\frac{m\pi}{L}}{\frac{\alpha_{1n}^2}{a^2} + \frac{m^2\pi^2}{L^2}} \exp\left[-\frac{\mu}{\rho}\left(\frac{\alpha_{1n}^2}{a^2} + \frac{m^2\pi^2}{L^2}\right)t\right] \frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})} \sin \frac{m\pi z}{L} \end{aligned}$$

Consequently, since $v_\theta(r, z, t) = v_\infty(r, z) - v_t(r, z, t)$,

$$\begin{aligned} v_\theta(r, z, t) &= \sum_{n=1}^{\infty} \frac{2\Omega a}{\alpha_{1n}} \frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})} \frac{\sinh\left[\frac{\alpha_{1n}}{a}(L-z)\right]}{\sinh\left(\frac{\alpha_{1n}}{a}L\right)} \\ &\quad - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{2}{L} \frac{2\Omega a}{\alpha_{1n}} \frac{\frac{m\pi}{L}}{\frac{\alpha_{1n}^2}{a^2} + \frac{m^2\pi^2}{L^2}} \exp\left[-\frac{\mu}{\rho}\left(\frac{\alpha_{1n}^2}{a^2} + \frac{m^2\pi^2}{L^2}\right)t\right] \frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})} \sin \frac{m\pi z}{L}. \end{aligned}$$

Therefore, for $t > 0$,

$$\boxed{v_\theta(r, z, t) = \sum_{n=1}^{\infty} \frac{2\Omega a}{\alpha_{1n}} \frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})} \left\{ \frac{\sinh\left[\frac{\alpha_{1n}}{a}(L-z)\right]}{\sinh\left(\frac{\alpha_{1n}}{a}L\right)} - \frac{2}{L} \sum_{m=1}^{\infty} \frac{\frac{m\pi}{L}}{\frac{\alpha_{1n}^2}{a^2} + \frac{m^2\pi^2}{L^2}} \exp\left[-\frac{\mu}{\rho}\left(\frac{\alpha_{1n}^2}{a^2} + \frac{m^2\pi^2}{L^2}\right)t\right] \sin \frac{m\pi z}{L} \right\}.}$$

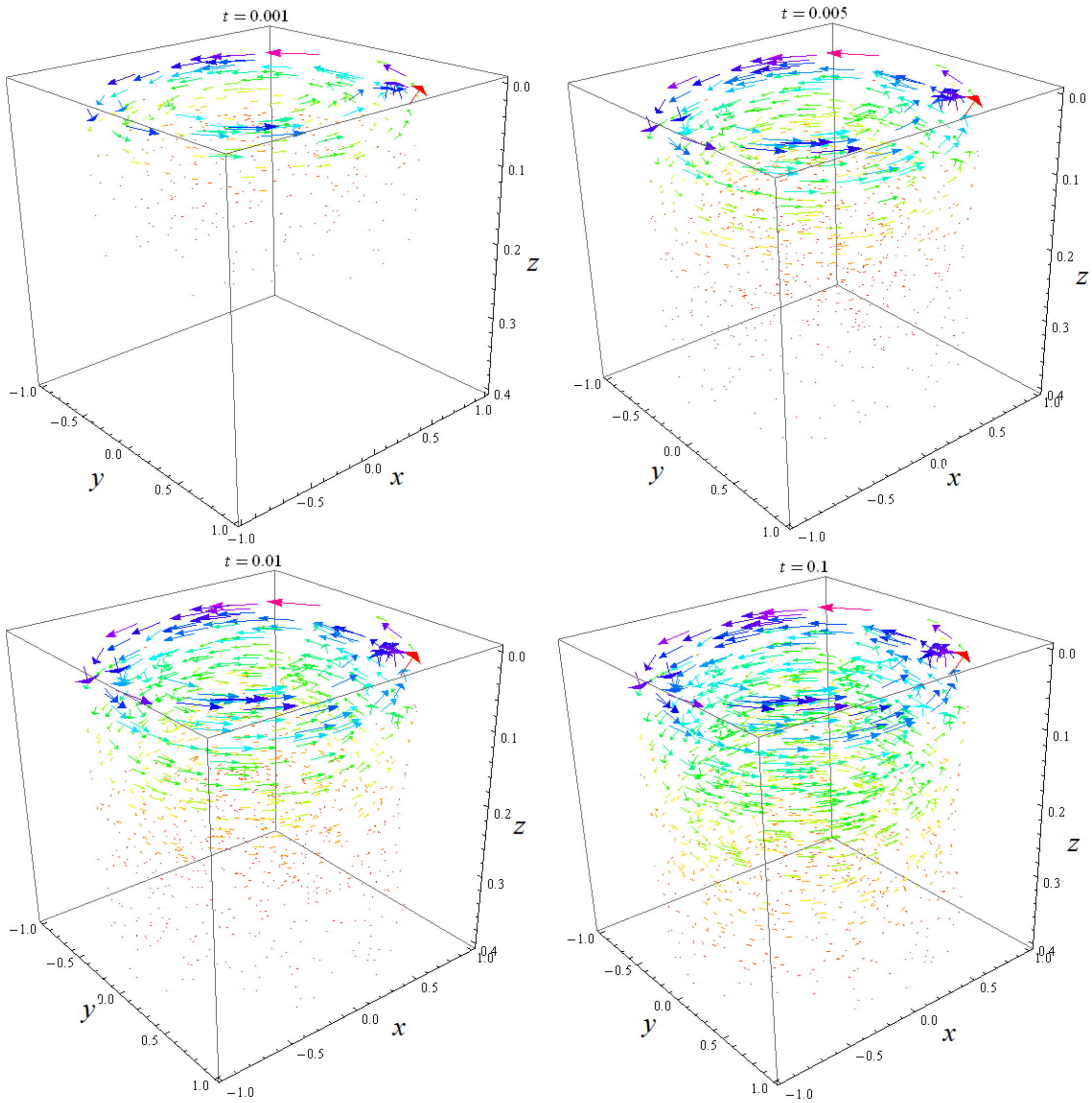

Figure 2: This figure shows the time evolution of the velocity field **v**. Snapshots are shown at $t = 0.001$, $t = 0.005$, $t = 0.01$, and $t = 0.1$ with $\Omega = 1$, $a = 1$, $L = 0.4$, $\mu = 1$, and $\rho = 1$. The fields are only approximate, as only the first 20 terms in the two infinite series have been used. As $t \to \infty$, the velocity field looks more and more like the one in Figure 1.

In preparation of taking the limit of $v_\theta(r, z, t)$ as $L \to \infty$, let $s_m = m\pi/L$ in the summand. Then

$$\Delta s = s_{m+1} - s_m = \frac{\pi}{L}$$

and

$$\frac{2}{L} = \frac{2}{\pi}\Delta s.$$

The boxed result for $v_\theta(r, z, t)$ becomes

$$v_\theta(r,z,t) = \sum_{n=1}^{\infty} \frac{2\Omega a}{\alpha_{1n}} \frac{J_1\left(\frac{\alpha_{1n}}{a} r\right)}{J_0(\alpha_{1n})} \left\{ \frac{\sinh\left[\frac{\alpha_{1n}}{a}(L-z)\right]}{\sinh\left(\frac{\alpha_{1n}}{a} L\right)} - \frac{2}{\pi} \sum_{m=1}^{\infty} \Delta s \frac{s_m}{\frac{\alpha_{1n}^2}{a^2} + s_m^2} \exp\left[-\frac{\mu}{\rho}\left(\frac{\alpha_{1n}^2}{a^2} + s_m^2\right) t\right] \sin s_m z \right\}.$$

In the limit as $L \to \infty$ the ratio of hyperbolic sines tends to an exponential function as shown at the end of part (a). In addition, Δs becomes an infinitesimal quantity ds, resulting in the sum's turning into an integral. The interval of integration is $(0, L)$, or $(0, \infty)$ in the limit. Therefore,

$$\boxed{\lim_{L\to\infty} v_\theta(r,z,t) = \sum_{n=1}^{\infty} \frac{2\Omega a}{\alpha_{1n}} \frac{J_1\left(\frac{\alpha_{1n}}{a} r\right)}{J_0(\alpha_{1n})} \left\{ \exp\left(-\frac{\alpha_{1n}}{a} z\right) - \frac{2}{\pi} \int_0^{\infty} ds\, \frac{s}{\frac{\alpha_{1n}^2}{a^2} + s^2} \exp\left[-\frac{\mu}{\rho}\left(\frac{\alpha_{1n}^2}{a^2} + s^2\right) t\right] \sin sz \right\}.}$$

Part (c) - A Tube of Finite Length

The situation is the same as in part (b) except now the boundary condition at $z = 0$ has changed. The governing PDE for the tangential velocity v_θ is

$$\rho \frac{\partial v_\theta}{\partial t} = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r}(r v_\theta) \right) + \frac{\partial^2 v_\theta}{\partial z^2} \right],$$

and the boundary conditions associated with it are

$$\begin{aligned} v_\theta(0, z, t) &= 0 \\ v_\theta(a, z, t) &= 0 \\ v_\theta(r, 0, t) &= -\Omega_0 r \Re\{e^{i\omega t}\} \\ v_\theta(r, L, t) &= 0, \end{aligned}$$

where $\Re\{e^{i\omega t}\}$ represents the real part of $e^{i\omega t}$. Here ω denotes the angular frequency that the disk oscillates. Since we only care about the oscillatory steady state, we assume a solution of the form

$$v_\theta(r, z, t) = \Re\{v^{\text{o}}(r, z) e^{i\omega t}\}$$

and plug it into the PDE.

$$\rho \frac{\partial}{\partial t} \Re\{v^{\text{o}}(r, z) e^{i\omega t}\} = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \Re\{v^{\text{o}}(r, z) e^{i\omega t}\}) \right) + \frac{\partial^2}{\partial z^2} \Re\{v^{\text{o}}(r, z) e^{i\omega t}\} \right],$$

$$\Re\left\{ i\omega\rho v^{\text{o}}(r, z) e^{i\omega t} \right\} = \Re\left\{ \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r}(r v^{\text{o}}) \right) + \frac{\partial^2 v^{\text{o}}}{\partial z^2} \right] e^{i\omega t} \right\}.$$

Use the fact that if $\Re\{z_1 w\} = \Re\{z_2 w\}$, where z_1, z_2, and w are complex, then $z_1 = z_2$.

$$i\omega\rho v^{\text{o}} = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r}(r v^{\text{o}}) \right) + \frac{\partial^2 v^{\text{o}}}{\partial z^2} \right]$$

This is a linear homogeneous PDE for v^{o} with boundary conditions,

$$\begin{aligned} v^{\text{o}}(0, z) &= 0 \\ v^{\text{o}}(a, z) &= 0 \\ v^{\text{o}}(r, 0) &= -\Omega_0 r \\ v^{\text{o}}(r, L) &= 0, \end{aligned}$$

which can be solved with the method of separation of variables. Assume a product solution of the form $v^{\text{o}}(r, z) = R(r)Z(z)$ and substitute it into the PDE

$$i\omega\rho R(r)Z(z) = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r}[rR(r)Z(z)] \right) + \frac{\partial^2}{\partial z^2}[R(r)Z(z)] \right]$$

$$\rightarrow \quad i\omega\rho RZ = \mu \left[Z \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr}(rR) \right) + RZ'' \right]$$

and the boundary conditions.

$$\begin{aligned} v^{\text{o}}(0, z) &= 0 & &\rightarrow & R(0)Z(z) &= 0 & &\rightarrow & R(0) &= 0 \\ v^{\text{o}}(a, z) &= 0 & &\rightarrow & R(a)Z(z) &= 0 & &\rightarrow & R(a) &= 0 \\ v^{\text{o}}(r, L) &= 0 & &\rightarrow & R(r)Z(L) &= 0 & &\rightarrow & Z(L) &= 0 \end{aligned}$$

Now separate variables in the PDE: divide both sides by $\mu R Z$

$$\frac{i\omega\rho}{\mu} = \frac{1}{R}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rR)\right) + \frac{Z''}{Z}$$

and then bring Z''/Z to the left side so that all functions of z are on the left side and all functions of r are on the right side. The final answer will be the same regardless of which side the constants are on.

$$\frac{i\omega\rho}{\mu} - \frac{Z''}{Z} = \frac{1}{R}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rR)\right)$$

The only way a function of z can be equal to a function of r is if both are equal to a constant λ.

$$\frac{i\omega\rho}{\mu} - \frac{Z''}{Z} = \frac{1}{R}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rR)\right) = \lambda$$

The eigenvalue problem for $R(r)$ was already solved in part (a). There we found that $\lambda = -\gamma^2$, where $\gamma a = \alpha_{1n}$, and

$$R(r) = C_5 J_1(\gamma r) \quad \rightarrow \quad R_n(r) = J_1\left(\frac{\alpha_{1n}}{a}r\right), \quad n = 1, 2, \ldots.$$

Consequently, the equation for Z becomes

$$\frac{i\omega\rho}{\mu} - \frac{Z''}{Z} = -\gamma^2.$$

Solve it for Z''.

$$Z'' = \left(\gamma^2 + \frac{i\omega\rho}{\mu}\right) Z$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Z(z) = C_{17}\cosh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}z\right) + C_{18}\sinh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}z\right)$$

Apply the $Z(L) = 0$ boundary condition to determine one of the constants.

$$Z(L) = C_{17}\cosh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}L\right) + C_{18}\sinh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}L\right) = 0$$

Solve this equation for C_{17}.

$$C_{17} = -C_{18}\frac{\sinh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}L\right)}{\cosh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}L\right)}$$

As a result, the eigenfunctions are

$$\begin{aligned} Z(z) &= -C_{18}\frac{\sinh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}L\right)}{\cosh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}L\right)}\cosh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}z\right) + C_{18}\sinh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}z\right) \\ &= -C_{18}\frac{\sinh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}L\right)\cosh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}z\right) - \cosh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}L\right)\sinh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}z\right)}{\cosh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}L\right)} \end{aligned}$$

$$Z(z) = -C_{18} \frac{\sinh\left[\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}(L-z)\right]}{\cosh\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}L\right)}$$

$$= C_{19} \sinh\left[\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}(L-z)\right] \quad \rightarrow \quad Z_n(z) = \sinh\left[\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}}(L-z)\right], \quad n = 1, 2, \ldots.$$

According to the principle of superposition, the solution to the PDE for $v^{\mathrm{o}}(r, z)$ is a linear combination of all products $R_n(r)Z_n(z)$ over all the eigenvalues.

$$v^{\mathrm{o}}(r, z) = \sum_{n=1}^{\infty} D_n J_1\left(\frac{\alpha_{1n}}{a}r\right) \sinh\left[\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}}(L-z)\right]$$

Our task now is to use the inhomogeneous boundary condition at $z = 0$ to determine D_n.

$$v^{\mathrm{o}}(r, 0) = \sum_{n=1}^{\infty} D_n J_1\left(\frac{\alpha_{1n}}{a}r\right) \sinh\left(\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}}L\right) = -\Omega_0 r$$

Multiply both sides by $rJ_1\left(\frac{\alpha_{1m}}{a}r\right)$, where m is an integer.

$$\sum_{n=1}^{\infty} D_n \sinh\left(\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}}L\right) J_1\left(\frac{\alpha_{1n}}{a}r\right) J_1\left(\frac{\alpha_{1m}}{a}r\right) r = -\Omega_0 r^2 J_1\left(\frac{\alpha_{1m}}{a}r\right)$$

Integrate both sides with respect to r from 0 to a.

$$\int_0^a \sum_{n=1}^{\infty} D_n \sinh\left(\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}}L\right) J_1\left(\frac{\alpha_{1n}}{a}r\right) J_1\left(\frac{\alpha_{1m}}{a}r\right) r\,dr = -\int_0^a \Omega_0 r^2 J_1\left(\frac{\alpha_{1m}}{a}r\right) dr$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} D_n \sinh\left(\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}}L\right) \int_0^a J_1\left(\frac{\alpha_{1n}}{a}r\right) J_1\left(\frac{\alpha_{1m}}{a}r\right) r\,dr = -\Omega_0 \int_0^a r^2 J_1\left(\frac{\alpha_{1m}}{a}r\right) dr$$

Because the Bessel functions are orthogonal with one another on $[0, a]$ with respect to the weight r, the integral on the left side is zero for $n \neq m$. The $n = m$ term is all that remains in the infinite series.

$$D_n \sinh\left(\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}}L\right) \int_0^a J_1^2\left(\frac{\alpha_{1n}}{a}r\right) r\,dr = -\Omega_0 \int_0^a r^2 J_1\left(\frac{\alpha_{1n}}{a}r\right) dr$$

Both of these integrals are known.

$$D_n \sinh\left(\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}}L\right) \cdot \frac{a^2}{2} J_2^2(\alpha_{1n}) = -\Omega_0 \cdot \frac{a^3}{\alpha_{1n}} J_2(\alpha_{1n})$$

Solve the equation for D_n.

$$D_n = -\frac{2\Omega_0 a}{\alpha_{1n}\sinh\left(\sqrt{\frac{\alpha_{1n}^2}{a^2}+\frac{i\omega\rho}{\mu}}L\right)J_2(\alpha_{1n})}$$

Here we apply a known property of Bessel functions, $J_{p+1}(x) = \frac{2p}{x}J_p(x) - J_{p-1}(x)$, to write J_2 in terms of J_1 and J_0.

$$D_n = -\frac{2\Omega_0 a}{\alpha_{1n}\sinh\left(\sqrt{\frac{\alpha_{1n}^2}{a^2}+\frac{i\omega\rho}{\mu}}L\right)\left[\frac{2}{\alpha_{1n}}\underbrace{J_1(\alpha_{1n})}_{=0}-J_0(\alpha_{1n})\right]} = \frac{2\Omega_0 a}{\alpha_{1n}\sinh\left(\sqrt{\frac{\alpha_{1n}^2}{a^2}+\frac{i\omega\rho}{\mu}}L\right)J_0(\alpha_{1n})}$$

We have for v^{o} then

$$v^{\text{o}}(r,z) = \sum_{n=1}^{\infty}\frac{2\Omega_0 a}{\alpha_{1n}\sinh\left(\sqrt{\frac{\alpha_{1n}^2}{a^2}+\frac{i\omega\rho}{\mu}}L\right)J_0(\alpha_{1n})}J_1\left(\frac{\alpha_{1n}}{a}r\right)\sinh\left[\sqrt{\frac{\alpha_{1n}^2}{a^2}+\frac{i\omega\rho}{\mu}}(L-z)\right].$$

Thus,

$$v^{\text{o}}(r,z) = \sum_{n=1}^{\infty}\frac{2\Omega_0 a}{\alpha_{1n}}\frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})}\frac{\sinh\left[\sqrt{\frac{\alpha_{1n}^2}{a^2}+\frac{i\omega\rho}{\mu}}(L-z)\right]}{\sinh\left(\sqrt{\frac{\alpha_{1n}^2}{a^2}+\frac{i\omega\rho}{\mu}}L\right)}.$$

Suppose that the ratio of hyperbolic sines can be written as a complex function $F_1(z) + iF_2(z)$, where F_1 and F_2 are real functions of z. Then

$$\begin{aligned}
v_\theta(r,z,t) &= \Re\{v^{\text{o}}(r,z)e^{i\omega t}\} \\
&= \Re\left\{\sum_{n=1}^{\infty}\frac{2\Omega_0 a}{\alpha_{1n}}\frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})}[F_1(z)+iF_2(z)](\cos\omega t + i\sin\omega t)\right\} \\
&= \sum_{n=1}^{\infty}\frac{2\Omega_0 a}{\alpha_{1n}}\frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})}\Re\left\{[F_1(z)\cos\omega t - F_2(z)\sin\omega t] + i[F_1(z)\sin\omega t + F_2(z)\cos\omega t]\right\},
\end{aligned}$$

and the oscillatory steady-state solution is therefore

$$\boxed{v_\theta(r,z,t) = \sum_{n=1}^{\infty}\frac{2\Omega_0 a}{\alpha_{1n}}\frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})}[F_1(z)\cos\omega t - F_2(z)\sin\omega t].}$$

The aim now is to find $F_1(z)$ and $F_2(z)$.

$$F_1(z) + iF_2(z) = \frac{\sinh\left[\sqrt{\frac{\alpha_{1n}^2}{a^2}+\frac{i\omega\rho}{\mu}}(L-z)\right]}{\sinh\left(\sqrt{\frac{\alpha_{1n}^2}{a^2}+\frac{i\omega\rho}{\mu}}L\right)}$$

Suppose further that the square root can be represented as a complex number $E_1 + iE_2$, where E_1 and E_2 are real constants.

$$= \frac{\sinh[(E_1+iE_2)(L-z)]}{\sinh[(E_1+iE_2)L]}$$

Expand the arguments.

$$F_1(z) + iF_2(z) = \frac{\sinh[E_1(L-z) + iE_2(L-z)]}{\sinh(E_1L + iE_2L)}$$

Use the angle addition formula for hyperbolic sine.

$$= \frac{\sinh[E_1(L-z)]\cosh[iE_2(L-z)] + \cosh[E_1(L-z)]\sinh[iE_2(L-z)]}{\sinh(E_1L)\cosh(iE_2L) + \cosh(E_1L)\sinh(iE_2L)}$$

Use the facts that $\sinh is = i\sin s$ and $\cosh is = \cos s$.

$$= \frac{\sinh[E_1(L-z)]\cos[E_2(L-z)] + i\cosh[E_1(L-z)]\sin[E_2(L-z)]}{\sinh(E_1L)\cos(E_2L) + i\cosh(E_1L)\sin(E_2L)}$$

Now multiply the numerator and denominator by the complex conjugate.

$$= \frac{\sinh[E_1(L-z)]\cos[E_2(L-z)] + i\cosh[E_1(L-z)]\sin[E_2(L-z)]}{\sinh(E_1L)\cos(E_2L) + i\cosh(E_1L)\sin(E_2L)} \times \frac{\sinh(E_1L)\cos(E_2L) - i\cosh(E_1L)\sin(E_2L)}{\sinh(E_1L)\cos(E_2L) - i\cosh(E_1L)\sin(E_2L)}$$

Multiplying the fractions, simplifying the result, and matching the real and imaginary parts on both sides of the equation, we have

$$\boxed{F_1(z) = \frac{\cos(E_2z)\cosh[E_1(2L-z)] - \cos[E_2(2L-z)]\cosh(E_1z)}{\cosh(2E_1L) - \cos(2E_2L)}}$$

and

$$\boxed{F_2(z) = -\frac{\sin(E_2z)\sinh[E_1(2L-z)] - \sin[E_2(2L-z)]\sinh(E_1z)}{\cosh(2E_1L) - \cos(2E_2L)}}.$$

The final task is to find E_1 and E_2.

$$E_1 + iE_2 = \sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}}$$

In general the square root of a complex number w is a double-valued function, which can be written as

$$w^{1/2} = \exp\left(\frac{1}{2}\log w\right).$$

The principal branch of $w^{1/2}$ is thus obtained by taking the principal branch of $\log w$.

$$\begin{aligned} &= \exp\left(\frac{1}{2}\operatorname{Log} w\right), \quad (|w| > 0,\ -\pi < \operatorname{Arg} w < \pi) \\ &= \exp\left[\frac{1}{2}(\ln M + i\Theta)\right] \\ &= \sqrt{M}e^{i\Theta/2}, \end{aligned}$$

where $M = |w|$ is the magnitude of w and $\Theta = \operatorname{Arg} w$ is the principal argument of w. In other words, this expression for $\sqrt{w}$ can be used for a complex number $w = Me^{i\Theta}$. So then

$$\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}} = \sqrt{\sqrt{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}} \exp\left[\frac{i}{2}\tan^{-1}\left(\frac{\frac{\omega\rho}{\mu}}{\frac{\alpha_{1n}^2}{a^2}}\right)\right]}$$

$$= \sqrt[4]{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}} \exp\left[\frac{i}{2}\tan^{-1}\left(\frac{\omega\rho a^2}{\mu\alpha_{1n}^2}\right)\right]$$

Use Euler's formula.

$$= \sqrt[4]{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}} \left\{\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{\omega\rho a^2}{\mu\alpha_{1n}^2}\right)\right] + i\sin\left[\frac{1}{2}\tan^{-1}\left(\frac{\omega\rho a^2}{\mu\alpha_{1n}^2}\right)\right]\right\}$$

Use the half-angle formulas for sine and cosine.

$$= \sqrt[4]{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}} \left\{\sqrt{\frac{1+\cos\left[\tan^{-1}\left(\frac{\omega\rho a^2}{\mu\alpha_{1n}^2}\right)\right]}{2}} + i\sqrt{\frac{1-\cos\left[\tan^{-1}\left(\frac{\omega\rho a^2}{\mu\alpha_{1n}^2}\right)\right]}{2}}\right\}$$

$$= \frac{1}{\sqrt{2}}\sqrt[4]{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}} \left\{\sqrt{1+\cos\left[\tan^{-1}\left(\frac{\omega\rho a^2}{\mu\alpha_{1n}^2}\right)\right]} + i\sqrt{1-\cos\left[\tan^{-1}\left(\frac{\omega\rho a^2}{\mu\alpha_{1n}^2}\right)\right]}\right\}$$

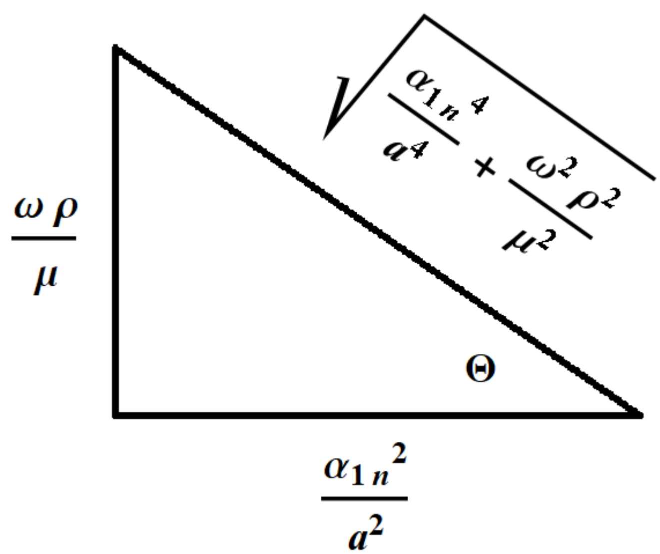

Figure 3: To determine the cosine of $\Theta = \tan^{-1}\left(\frac{\frac{\omega\rho}{\mu}}{\frac{\alpha_{1n}^2}{a^2}}\right)$, it helps to draw out the right triangle. The Pythagorean theorem is used to determine the hypotenuse.

$$\begin{aligned}
\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}} &= \frac{1}{\sqrt{2}}\sqrt[4]{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}}\left(\sqrt{1 + \frac{\frac{\alpha_{1n}^2}{a^2}}{\sqrt{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}}}} + i\sqrt{1 - \frac{\frac{\alpha_{1n}^2}{a^2}}{\sqrt{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}}}}\right) \\
&= \frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}} + \frac{\alpha_{1n}^2}{a^2}} + i\sqrt{\sqrt{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}} - \frac{\alpha_{1n}^2}{a^2}}\right) \\
&= \frac{1}{\sqrt{2}}\sqrt{\sqrt{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}} + \frac{\alpha_{1n}^2}{a^2}} + i\frac{1}{\sqrt{2}}\sqrt{\sqrt{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}} - \frac{\alpha_{1n}^2}{a^2}} \\
&= E_1 + iE_2
\end{aligned}$$

Therefore,

$$\boxed{E_1 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}} + \frac{\alpha_{1n}^2}{a^2}}}$$

and

$$\boxed{E_2 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\frac{\alpha_{1n}^4}{a^4} + \frac{\omega^2\rho^2}{\mu^2}} - \frac{\alpha_{1n}^2}{a^2}}.}$$

Notice that if $\omega = 0$, then the answer simplifies to the first boxed solution in part (a).

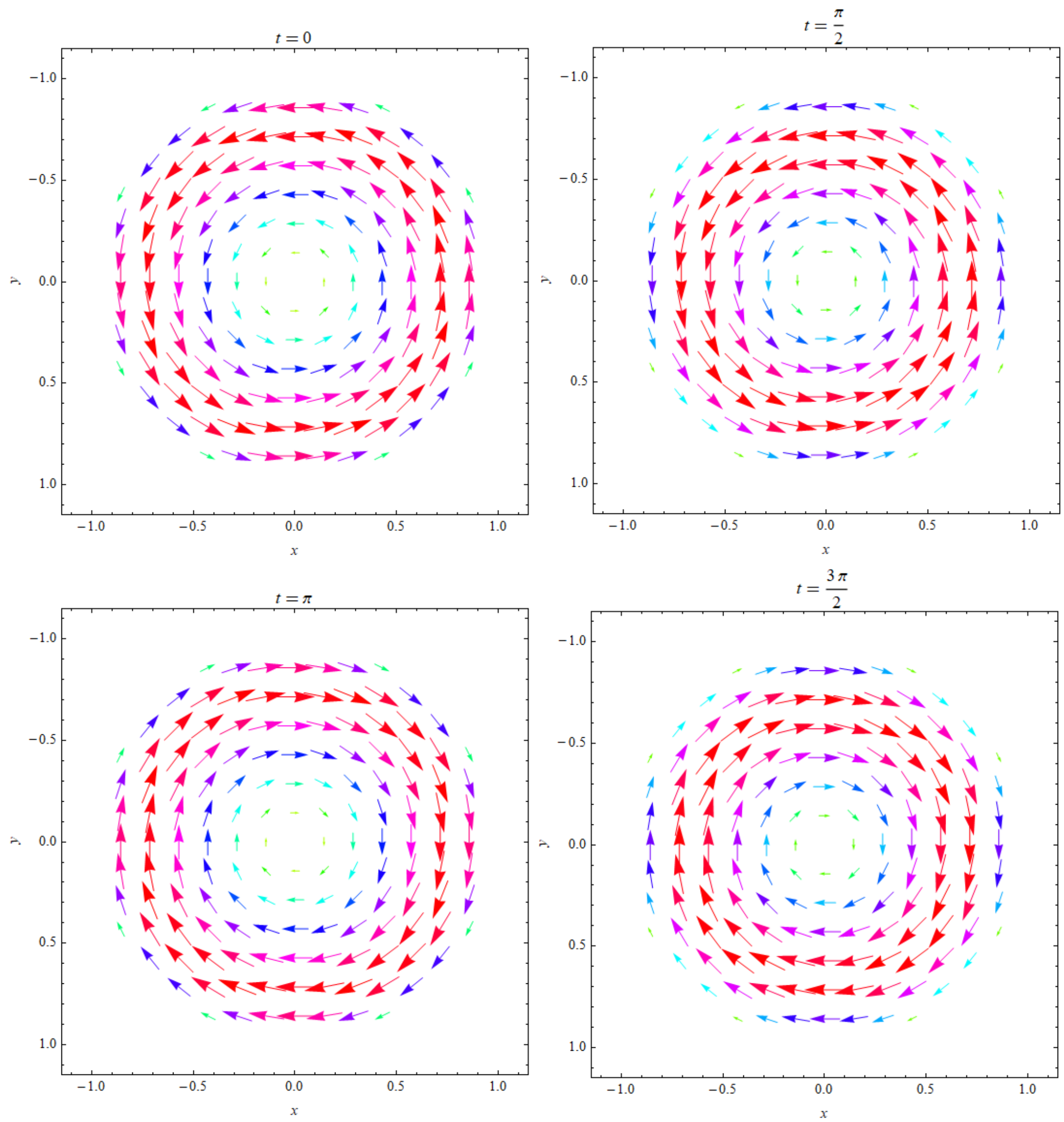

Figure 4: This figure shows the cross-sectional velocity field $\mathbf{v}$ at $z = 0.1$ (from the top looking down the tube) when $t = 0$, $t = \pi/2$, $t = \pi$, and $t = 3\pi/2$ with $\Omega_0 = 1$, $a = 1$, $L = 0.4$, $\mu = 1$, $\rho = 1$, and $\omega = 1$. The fields are only approximate, as only the first 20 terms in the infinite series have been used. Notice that the velocity field changes direction at $t = \pi$ as a result of the oscillating disk. The field will be the same at $t = 2\pi$ as it was at $t = 0$ because sine and cosine are 2π-periodic.

Part (c) - A Tube of Infinite Length

The situation is the same as in part (c) except now $0 < z < \infty$. The governing PDE for the tangential velocity v_θ is

$$\rho \frac{\partial v_\theta}{\partial t} = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{\partial^2 v_\theta}{\partial z^2} \right],$$

and the boundary conditions associated with it are

$$\begin{aligned} v_\theta(0, z, t) &= 0 \\ v_\theta(a, z, t) &= 0 \\ v_\theta(r, 0, t) &= -\Omega_0 r \Re\{e^{i\omega t}\} \\ v_\theta(r, \infty, t) &= 0, \end{aligned}$$

where $\Re\{e^{i\omega t}\}$ represents the real part of $e^{i\omega t}$. Here ω denotes the angular frequency that the disk oscillates. Since we only care about the oscillatory steady state, we assume a solution of the form

$$v_\theta(r, z, t) = \Re\{v^{\text{o}}(r, z) e^{i\omega t}\}$$

and plug it into the PDE.

$$\begin{aligned} \rho \frac{\partial}{\partial t} \Re\{v^{\text{o}}(r, z) e^{i\omega t}\} &= \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \Re\{v^{\text{o}}(r, z) e^{i\omega t}\}) \right) + \frac{\partial^2}{\partial z^2} \Re\{v^{\text{o}}(r, z) e^{i\omega t}\} \right], \\ \Re\left\{ i\omega\rho v^{\text{o}}(r, z) e^{i\omega t} \right\} &= \Re\left\{ \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v^{\text{o}}) \right) + \frac{\partial^2 v^{\text{o}}}{\partial z^2} \right] e^{i\omega t} \right\}. \end{aligned}$$

Use the fact that if $\Re\{z_1 w\} = \Re\{z_2 w\}$, where z_1, z_2, and w are complex, then $z_1 = z_2$.

$$i\omega\rho v^{\text{o}} = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v^{\text{o}}) \right) + \frac{\partial^2 v^{\text{o}}}{\partial z^2} \right] \tag{5}$$

This is a linear homogeneous PDE for v^{o} with boundary conditions,

$$\begin{aligned} v^{\text{o}}(0, z) &= 0 \\ v^{\text{o}}(a, z) &= 0 \\ v^{\text{o}}(r, 0) &= -\Omega_0 r \\ v^{\text{o}}(r, \infty) &= 0, \end{aligned}$$

which can be solved with the method of separation of variables. Assume a product solution of the form $v^{\text{o}}(r, z) = R(r) Z(z)$ and substitute it into the PDE

$$i\omega\rho R(r) Z(z) = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} [r R(r) Z(z)] \right) + \frac{\partial^2}{\partial z^2} [R(r) Z(z)] \right]$$

$$\rightarrow \quad i\omega\rho RZ = \mu \left[Z \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (rR) \right) + RZ'' \right]$$

and the boundary conditions.

$$\begin{aligned} v^{\text{o}}(0, z) = 0 \quad &\rightarrow \quad R(0) Z(z) = 0 \quad \rightarrow \quad R(0) = 0 \\ v^{\text{o}}(a, z) = 0 \quad &\rightarrow \quad R(a) Z(z) = 0 \quad \rightarrow \quad R(a) = 0 \\ v^{\text{o}}(r, \infty) = 0 \quad &\rightarrow \quad R(r) Z(\infty) = 0 \quad \rightarrow \quad Z(\infty) = 0 \end{aligned}$$

Now separate variables in the PDE: divide both sides by μRZ

$$\frac{i\omega\rho}{\mu} = \frac{1}{R}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rR)\right) + \frac{Z''}{Z}$$

and then bring Z''/Z to the left side so that all functions of z are on the left side and all functions of r are on the right side. The final answer will be the same regardless of which side the constants are on.

$$\frac{i\omega\rho}{\mu} - \frac{Z''}{Z} = \frac{1}{R}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rR)\right)$$

The only way a function of z can be equal to a function of r is if both are equal to a constant λ.

$$\frac{i\omega\rho}{\mu} - \frac{Z''}{Z} = \frac{1}{R}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rR)\right) = \lambda$$

The eigenvalue problem for $R(r)$ was already solved in part (a). There we found that $\lambda = -\gamma^2$, where $\gamma a = \alpha_{1n}$, and

$$R(r) = C_5 J_1(\gamma r) \quad \rightarrow \quad R_n(r) = J_1\left(\frac{\alpha_{1n}}{a}r\right), \quad n = 1, 2, \ldots.$$

Consequently, the equation for Z becomes

$$\frac{i\omega\rho}{\mu} - \frac{Z''}{Z} = -\gamma^2.$$

Solve it for Z''.

$$Z'' = \left(\gamma^2 + \frac{i\omega\rho}{\mu}\right)Z$$

The general solution can be written in terms of exponential functions.

$$Z(z) = C_{19}\exp\left(\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}z\right) + C_{20}\exp\left(-\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}z\right)$$

In order for $Z(\infty) = 0$ to be satisfied, we require $C_{19} = 0$. So then

$$Z(z) = C_{20}\exp\left(-\sqrt{\gamma^2 + \frac{i\omega\rho}{\mu}}z\right) \quad \rightarrow \quad Z_n(z) = \exp\left(-\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}}z\right), \quad n = 1, 2, \ldots.$$

According to the principle of superposition, the solution to the PDE for $v^{\text{o}}(r, z)$ is a linear combination of all products $R_n(r)Z_n(z)$ over all the eigenvalues.

$$v^{\text{o}}(r, z) = \sum_{n=1}^{\infty} G_n \exp\left(-\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}}z\right) J_1\left(\frac{\alpha_{1n}}{a}r\right)$$

Now the inhomogeneous boundary condition at $z = 0$ will be used to determine the coefficients G_n.

$$v^{\text{o}}(r, 0) = \sum_{n=1}^{\infty} G_n J_1\left(\frac{\alpha_{1n}}{a}r\right) = -\Omega_0 r$$

Multiply both sides by $rJ_1\left(\frac{\alpha_{1m}}{a}r\right)$, where m is an integer.

$$\sum_{n=1}^{\infty} G_n J_1\left(\frac{\alpha_{1n}}{a}r\right) J_1\left(\frac{\alpha_{1m}}{a}r\right) r = -\Omega_0 r^2 J_1\left(\frac{\alpha_{1m}}{a}r\right)$$

Integrate both sides with respect to r from 0 to a.

$$\int_0^a \sum_{n=1}^{\infty} G_n J_1\left(\frac{\alpha_{1n}}{a}r\right) J_1\left(\frac{\alpha_{1m}}{a}r\right) r\,dr = -\int_0^a \Omega_0 r^2 J_1\left(\frac{\alpha_{1m}}{a}r\right) dr$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} G_n \int_0^a J_1\left(\frac{\alpha_{1n}}{a}r\right) J_1\left(\frac{\alpha_{1m}}{a}r\right) r\,dr = -\Omega_0 \int_0^a r^2 J_1\left(\frac{\alpha_{1m}}{a}r\right) dr$$

Because the Bessel functions are orthogonal with one another on $[0, a]$ with respect to the weight r, the integral on the left side is zero for $n \neq m$. The $n = m$ term is all that remains in the infinite series.

$$G_n \int_0^a J_1^2\left(\frac{\alpha_{1n}}{a}r\right) r\,dr = -\Omega_0 \int_0^a r^2 J_1\left(\frac{\alpha_{1n}}{a}r\right) dr$$

Both of these integrals are known.

$$G_n \cdot \frac{a^2}{2} J_2^2(\alpha_{1n}) = -\Omega_0 \cdot \frac{a^3}{\alpha_{1n}} J_2(\alpha_{1n})$$

Solve the equation for G_n.

$$G_n = -\frac{2\Omega_0 a}{\alpha_{1n} J_2(\alpha_{1n})}$$

Here we apply a known property of Bessel functions, $J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$, to write J_2 in terms of J_1 and J_0.

$$G_n = -\frac{2\Omega_0 a}{\alpha_{1n}\left[\frac{2}{\alpha_{1n}} \underbrace{J_1(\alpha_{1n})}_{=\,0} - J_0(\alpha_{1n})\right]} = \frac{2\Omega_0 a}{\alpha_{1n} J_0(\alpha_{1n})}$$

We have for v^{o} then

$$v^{\text{o}}(r, z) = \sum_{n=1}^{\infty} \frac{2\Omega_0 a}{\alpha_{1n} J_0(\alpha_{1n})} \exp\left(-\sqrt{\frac{\alpha_{1n}^2}{a^2} + \frac{i\omega\rho}{\mu}}\, z\right) J_1\left(\frac{\alpha_{1n}}{a}r\right) = \sum_{n=1}^{\infty} \frac{2\Omega_0 a}{\alpha_{1n}} \frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})} e^{-(E_1 + iE_2)z}.$$

Consequently,

$$\begin{aligned} v_\theta(r, z, t) &= \Re\{v^{\text{o}}(r, z) e^{i\omega t}\} \\ &= \Re\left\{\sum_{n=1}^{\infty} \frac{2\Omega_0 a}{\alpha_{1n}} \frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})} e^{-(E_1 + iE_2)z} e^{i\omega t}\right\} \\ &= \sum_{n=1}^{\infty} \frac{2\Omega_0 a}{\alpha_{1n}} \frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})} e^{-E_1 z} \Re\left\{e^{i(\omega t - E_2 z)}\right\}. \end{aligned}$$

Therefore, the oscillatory steady-state solution for a tube of infinite length is

$$\boxed{v_\theta(r, z, t) = \sum_{n=1}^{\infty} \frac{2\Omega_0 a}{\alpha_{1n}} \frac{J_1\left(\frac{\alpha_{1n}}{a}r\right)}{J_0(\alpha_{1n})} e^{-E_1 z} \cos(\omega t - E_2 z).}$$

Notice that if $\omega = 0$, then this result reduces to the second boxed equation in part (a).