Problem 4D.4

Unsteady annular flows.

(a) Obtain a solution to the Navier-Stokes equation for the start-up of *axial* annular flow by a sudden impressed pressure gradient. Check your result against the published solution.[10]

(b) Solve the Navier-Stokes equation for the unsteady *tangential* flow in an annulus. The fluid is at rest for $t < 0$. Starting at $t = 0$ the outer cylinder begins rotating with a constant angular velocity to cause laminar flow for $t > 0$. Compare your result with the published solution.[11]

Solution

Part (a)

Since the flow is unsteady and occuring along the axis of an annular tube, we assume that the velocity varies as a function of radius and time and that the fluid moves only in the z-direction.

$$\mathbf{v} = v_z(r,t)\hat{\mathbf{z}}$$

If we assume the fluid does not slip on the tube wall, then it has the wall's velocity at the inner radius ($r = \kappa R$) and the outer radius ($r = R$).

$$\begin{aligned} \text{Boundary Condition 1:} \quad & v_z(R,t) = 0 \\ \text{Boundary Condition 2:} \quad & v_z(\kappa R,t) = 0 \end{aligned}$$

The fluid starts from rest, so the initial velocity is zero.

$$\text{Initial Condition:} \quad v_z(r,0) = 0$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. If the fluid density ρ is constant, then the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming that ρ and the fluid viscosity μ are constant, it simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using cylindrical coordinates is the appropriate choice for this problem, so the two previous equations will be used in (r, θ, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

[10]W. Müller, *Zeits. für angew. Math. u. Mech.*, **16**, 227–238 (1936).
[11]R. B. Bird and C. F. Curtiss, *Chem. Engr. Sci*, **11**, 108–113 (1959).

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=0}+\underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0}+\underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0}+\underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0}-\underbrace{\frac{v_\theta^2}{r}}_{=0}\right)=-\underbrace{\frac{\partial p}{\partial r}}_{=0}+\mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0}+\underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0}+\underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0}-\underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\right]+\underbrace{\rho g_r}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0}+\underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0}+\underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0}+\underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0}+\underbrace{\frac{v_r v_\theta}{r}}_{=0}\right)=-\underbrace{\frac{1}{r}\frac{\partial p}{\partial \theta}}_{=0}+\mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)}_{=0}+\underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0}+\underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0}+\underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right]+\underbrace{\rho g_\theta}_{=0}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+\underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0}+\underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0}+\underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0}+\underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right]+\rho g_z$$

The relevant equation for the velocity is the z-equation, which has simplified considerably from the assumption that $\mathbf{v}=v_z\hat{\mathbf{z}}$.

$$\rho\frac{\partial v_z}{\partial t}=-\frac{\partial p}{\partial z}+\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\rho g_z$$

The sum of $-\partial p/\partial z$ and ρg_z is the impressed pressure gradient and will be denoted as $-(\mathscr{P}_L-\mathscr{P}_0)/(L-0)$.

$$\rho\frac{\partial v_z}{\partial t}=\frac{\mathscr{P}_0-\mathscr{P}_L}{L}+\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

The aim now is to put the partial differential equation into dimensionless form. Multiply both sides by $4L/(\mathscr{P}_0-\mathscr{P}_L)$.

$$\rho\frac{\partial v_z}{\partial t}\frac{4L}{\mathscr{P}_0-\mathscr{P}_L}=4+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\frac{4\mu L}{\mathscr{P}_0-\mathscr{P}_L}\right)$$

Introduce R^2 in the numerator and denominator on both sides.

$$\rho R^2\frac{\partial v_z}{\partial t}\frac{4L}{(\mathscr{P}_0-\mathscr{P}_L)R^2}=4+\frac{R}{r}R\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\frac{4\mu L}{(\mathscr{P}_0-\mathscr{P}_L)R^2}\right]$$

Introduce μ in the numerator and denominator on the left side, and introduce R in the numerator and denominator on the right side.

$$\frac{\rho R^2}{\mu}\frac{\partial v_z}{\partial t}\frac{4\mu L}{(\mathscr{P}_0-\mathscr{P}_L)R^2}=4+\frac{R}{r}R\frac{\partial}{\partial r}\left[\frac{r}{R}R\frac{\partial v_z}{\partial r}\frac{4\mu L}{(\mathscr{P}_0-\mathscr{P}_L)R^2}\right]$$

Here the dependent variable will be changed. Let

$$\phi=v_z\frac{4\mu L}{(\mathscr{P}_0-\mathscr{P}_L)R^2}.$$

The independent variables will also be changed to ones that are dimensionless.

$$\xi=\frac{r}{R}\quad\rightarrow\quad d\xi=\frac{dr}{R}\quad\Rightarrow\quad R\frac{\partial}{\partial r}=\frac{\partial}{\partial \xi}$$

$$\tau=\frac{\mu t}{\rho R^2}\quad\rightarrow\quad d\tau=\frac{\mu\, dt}{\rho R^2}\quad\Rightarrow\quad \frac{\rho R^2}{\mu}\frac{\partial}{\partial t}=\frac{\partial}{\partial \tau}$$

Therefore, the governing differential equation for velocity with dimensionless variables is

$$\frac{\partial \phi}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi}\right). \tag{1}$$

In terms of the new variables, the boundary and initial conditions become

$$\phi(1, \tau) = 0 \tag{2}$$

$$\phi(\kappa, \tau) = 0 \tag{3}$$

$$\phi(\xi, 0) = 0. \tag{4}$$

Due to the impressed pressure gradient, the fluid starts to move and eventually reaches a steady state. As a result, we expect the solution to be of the form $\phi(\xi, \tau) = \phi_\infty(\xi) - \phi_t(\xi, \tau)$, where ϕ_∞ is the steady-state velocity profile that is reached after a long time has passed and $\phi_t(\xi, \tau)$ is the transient velocity profile that dies out as τ increases. Substitute the expression for ϕ into equation (1).

$$\frac{\partial}{\partial \tau}[\phi_\infty(\xi) - \phi_t(\xi, \tau)] = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left[\xi \frac{\partial}{\partial \xi}[\phi_\infty(\xi) - \phi_t(\xi, \tau)]\right]$$

Distribute the operators on both sides.

$$-\frac{\partial \phi_t}{\partial \tau} = 4 + \frac{1}{\xi}\frac{d}{d\xi}\left(\xi \frac{d\phi_\infty}{d\xi}\right) - \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi_t}{\partial \xi}\right)$$

If we set

$$4 + \frac{1}{\xi}\frac{d}{d\xi}\left(\xi \frac{d\phi_\infty}{d\xi}\right) = 0, \tag{5}$$

then the previous equation reduces to

$$-\frac{\partial \phi_t}{\partial \tau} = -\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi_t}{\partial \xi}\right). \tag{6}$$

Solve equation (5) for the steady-state velocity profile.

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi \frac{d\phi_\infty}{d\xi}\right) = -4$$

Multiply both sides by ξ.

$$\frac{d}{d\xi}\left(\xi \frac{d\phi_\infty}{d\xi}\right) = -4\xi$$

Integrate both sides with respect to ξ.

$$\xi \frac{d\phi_\infty}{d\xi} = -2\xi^2 + C_1$$

Divide both sides by ξ.

$$\frac{d\phi_\infty}{d\xi} = -2\xi + \frac{C_1}{\xi}$$

Integrate both sides with respect to ξ once more.

$$\phi_\infty(\xi) = -\xi^2 + C_1 \ln \xi + C_2$$

The boundary conditions, $\phi(1,\tau)=0$ and $\phi(\kappa,\tau)=0$, hold for all time, including the steady state. Apply $\phi_\infty(1)=0$ and $\phi_\infty(\kappa)=0$ to determine C_1 and C_2.

$$\begin{aligned}\phi_\infty(1) &= -1 + C_2 = 0\\ \phi_\infty(\kappa) &= -\kappa^2 + C_1\ln\kappa + C_2 = 0\end{aligned}$$

Solving the system yields $C_1 = -(1-\kappa^2)/\ln\kappa$ and $C_2 = 1$. Thus,

$$\phi_\infty(\xi) = 1 - \xi^2 - (1-\kappa^2)\frac{\ln\xi}{\ln\kappa}.$$

Now we will solve equation (6) for ϕ_t. Multiply both sides of it by -1.

$$\frac{\partial\phi_t}{\partial\tau} = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi_t}{\partial\xi}\right)$$

To find the initial and boundary conditions associated with it, substitute $\phi(\xi,\tau)=\phi_\infty(\xi)-\phi_t(\xi,\tau)$ into equations (2), (3), and (4).

$$\begin{aligned}
\phi(1,\tau) &= \phi_\infty(1) - \phi_t(1,\tau) = 0 - \phi_t(1,\tau) = 0 &\quad\rightarrow\quad& \phi_t(1,\tau)=0\\
\phi(\kappa,\tau) &= \phi_\infty(\kappa) - \phi_t(\kappa,\tau) = 0 - \phi_t(\kappa,\tau) = 0 &\quad\rightarrow\quad& \phi_t(\kappa,\tau)=0\\
\phi(\xi,0) &= \phi_\infty(\xi) - \phi_t(\xi,0) = 0 &\quad\rightarrow\quad& \phi_t(\xi,0)=\phi_\infty(\xi)
\end{aligned}$$

The PDE for ϕ_t and its boundary conditions are linear and homogeneous, so the problem can be solved with the method of separation of variables. Assume a product solution of the form $\phi_t = X(\xi)T(\tau)$ and plug it into the PDE

$$XT' = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi X'T\right)$$

and the boundary conditions.

$$\begin{aligned}
\phi_t(1,\tau)=0 &\quad\rightarrow\quad X(1)T(\tau)=0 &\quad\rightarrow\quad& X(1)=0\\
\phi_t(\kappa,\tau)=0 &\quad\rightarrow\quad X(\kappa)T(\tau)=0 &\quad\rightarrow\quad& X(\kappa)=0
\end{aligned}$$

Now separate variables in the PDE: bring the functions of τ to the left side and bring the functions of ξ to the right side.

$$\frac{T'}{T} = \frac{1}{\xi X}\frac{d}{d\xi}\left(\xi X'\right)$$

The only way a function of τ can be equal to a function of ξ is if both are equal to a constant λ.

$$\frac{T'}{T} = \frac{1}{\xi X}\frac{d}{d\xi}\left(\xi X'\right) = \lambda$$

Values of λ for which the boundary conditions are satisfied are known as the eigenvalues, and the nontrivial functions associated with them are known as the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Assuming λ is positive, the differential equation for T becomes

$$\frac{T'}{T} = \mu^2.$$

Multiply both sides by T.

$$T' = \mu^2 T$$

The general solution is written in terms of the exponential function.

$$T(\tau) = C_3 e^{\mu^2 \tau}$$

The possibility that there are positive eigenvalues can be dismissed here because $T(\tau)$ diverges as $\tau \to \infty$.

Determination of the Zero Eigenvalue: $\lambda = 0$

Assuming λ is zero, the differential equation for X becomes

$$\frac{1}{\xi X}\frac{d}{d\xi}\left(\xi X'\right) = 0$$

Multiply both sides by ξX.

$$\frac{d}{d\xi}\left(\xi X'\right) = 0$$

Integrate both sides with respect to ξ.

$$\xi X' = C_4$$

Divide both sides by ξ.

$$X' = \frac{C_4}{\xi}$$

Integrate both sides with respect to ξ once more.

$$X(\xi) = C_4 \ln \xi + C_5$$

Apply the boundary conditions here to determine C_4 and C_5.

$$X(1) = C_5 = 0$$
$$X(\kappa) = C_4 \ln \kappa + C_5 = 0$$

Since $C_5 = 0$, the second equation gives $C_4 = 0$, which results in the trivial solution. Zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Assuming λ is negative, the differential equation for X becomes

$$\frac{1}{\xi X}\frac{d}{d\xi}\left(\xi X'\right) = -\gamma^2.$$

Multiply both sides by $\xi^2 X$.

$$\xi\frac{d}{d\xi}\left(\xi X'\right) = -\gamma^2 \xi^2 X$$

Expand the left side and bring $\gamma^2 \xi^2 X$ to the left.

$$\xi^2 X'' + \xi X' + \gamma^2 \xi^2 X = 0$$

This is the parametric form of Bessel's equation of order zero. The general solution is written in terms of zero-order Bessel functions of the first kind $J_0(\gamma\xi)$ and second kind $Y_0(\gamma\xi)$.

$$X(\xi) = C_6 J_0(\gamma\xi) + C_7 Y_0(\gamma\xi)$$

Apply the boundary conditions here to determine C_6 and C_7.

$$\begin{aligned} X(1) &= C_6 J_0(\gamma) + C_7 Y_0(\gamma) = 0 \\ X(\kappa) &= C_6 J_0(\gamma\kappa) + C_7 Y_0(\gamma\kappa) = 0 \end{aligned}$$

Solve the first equation for C_7

$$C_7 = -C_6 \frac{J_0(\gamma)}{Y_0(\gamma)}$$

and plug the result into the second equation.

$$C_6 J_0(\gamma\kappa) - C_6 \frac{J_0(\gamma)}{Y_0(\gamma)} Y_0(\gamma\kappa) = 0$$

To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Write the two terms on the left side as one.

$$C_6 \frac{J_0(\gamma\kappa)Y_0(\gamma) - J_0(\gamma)Y_0(\gamma\kappa)}{Y_0(\gamma)} = 0$$

Hence, γ is defined implicitly by $J_0(\gamma\kappa)Y_0(\gamma) - J_0(\gamma)Y_0(\gamma\kappa) = 0$. J_0 and Y_0 are oscillatory functions, so there are infinitely many values of γ. If γ_n denotes the nth zero of the function, then

$$\boxed{J_0(\gamma_n\kappa)Y_0(\gamma_n) - J_0(\gamma_n)Y_0(\gamma_n\kappa) = 0, \quad n = 1, 2, \ldots.}$$

The eigenfunctions associated with these eigenvalues for λ are

$$\begin{aligned} X(\xi) &= C_6 J_0(\gamma\xi) + C_7 Y_0(\gamma\xi) \\ &= C_6 J_0(\gamma\xi) - C_6 \frac{J_0(\gamma)}{Y_0(\gamma)} Y_0(\gamma\xi) \\ &= \frac{C_6}{Y_0(\gamma)}[J_0(\gamma\xi)Y_0(\gamma) - J_0(\gamma)Y_0(\gamma\xi)] \\ &= C_8[J_0(\gamma\xi)Y_0(\gamma) - J_0(\gamma)Y_0(\gamma\xi)] \quad \rightarrow \quad \boxed{X_n(\xi) = J_0(\gamma_n\xi)Y_0(\gamma_n) - J_0(\gamma_n)Y_0(\gamma_n\xi), \quad n = 1, 2, \ldots.} \end{aligned}$$

Now the ODE for T will be solved.

$$\frac{T'}{T} = -\gamma^2$$

Multiply both sides by T.

$$T' = -\gamma^2 T$$

The general solution is written in terms of the exponential function.

$$T(\tau) = C_9 e^{-\gamma^2\tau} \quad \rightarrow \quad T_n(\tau) = e^{-\gamma_n^2\tau}, \quad n = 1, 2, \ldots$$

According to the principle of superposition, the general solution to the PDE for ϕ_t is a linear combination of the eigenfunctions $X_n(\xi)T_n(\tau)$ over all the eigenvalues.

$$\phi_t(\xi, \tau) = \sum_{n=1}^{\infty} A_n e^{-\gamma_n^2\tau} X_n(\xi)$$

Apply the initial condition here to determine A_n.

$$\phi_t(\xi, 0) = \sum_{n=1}^{\infty} A_n X_n(\xi) = \phi_\infty(\xi)$$

Substitute the steady-state velocity distribution found for ϕ_∞.

$$\sum_{n=1}^{\infty} A_n X_n(\xi) = 1 - \xi^2 - (1-\kappa^2)\frac{\ln \xi}{\ln \kappa}$$

Multiply both sides by $X_m(\xi)\xi$, where m is an integer.

$$\sum_{n=1}^{\infty} A_n X_n(\xi) X_m(\xi)\xi = \left[1 - \xi^2 - (1-\kappa^2)\frac{\ln \xi}{\ln \kappa}\right] X_m(\xi)\xi$$

Integrate both sides with respect to ξ from κ to 1.

$$\int_\kappa^1 \sum_{n=1}^{\infty} A_n X_n(\xi) X_m(\xi)\xi \, d\xi = \int_\kappa^1 \left[1 - \xi^2 - (1-\kappa^2)\frac{\ln \xi}{\ln \kappa}\right] X_m(\xi)\xi \, d\xi$$

Bring the constants in front of the integral on the left side.

$$\sum_{n=1}^{\infty} A_n \int_\kappa^1 X_n(\xi) X_m(\xi)\xi \, d\xi = \int_\kappa^1 \left[1 - \xi^2 - (1-\kappa^2)\frac{\ln \xi}{\ln \kappa}\right] X_m(\xi)\xi \, d\xi$$

Because the $X_n(\xi)$ satisfy an ODE of the Sturm-Liouville form, they are guaranteed to be orthogonal with respect to the weight ξ, meaning that the integral on the left side is zero for $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$A_n \int_\kappa^1 X_n^2(\xi)\xi \, d\xi = \int_\kappa^1 \left[1 - \xi^2 - (1-\kappa^2)\frac{\ln \xi}{\ln \kappa}\right] X_n(\xi)\xi \, d\xi$$

Solve this equation for A_n.

$$\boxed{A_n = \frac{\displaystyle\int_\kappa^1 \left[1 - \xi^2 - (1-\kappa^2)\frac{\ln \xi}{\ln \kappa}\right] X_n(\xi)\xi \, d\xi}{\displaystyle\int_\kappa^1 X_n^2(\xi)\xi \, d\xi}}$$

The dimensionless velocity is then

$$\phi(\xi, \tau) = 1 - \xi^2 - (1-\kappa^2)\frac{\ln \xi}{\ln \kappa} - \sum_{n=1}^{\infty} A_n e^{-\gamma_n^2 \tau} X_n(\xi).$$

Changing back to the original variables, the unsteady velocity distribution for $t > 0$ in an annulus due to a sudden impressed pressure gradient is therefore

$$\boxed{v_z(r,t) = \frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2 - \frac{1-\kappa^2}{\ln(1/\kappa)}\ln\left(\frac{R}{r}\right) - \sum_{n=1}^{\infty} A_n \exp\left(-\gamma_n^2 \frac{\mu t}{\rho R^2}\right) X_n\left(\frac{r}{R}\right)\right].}$$

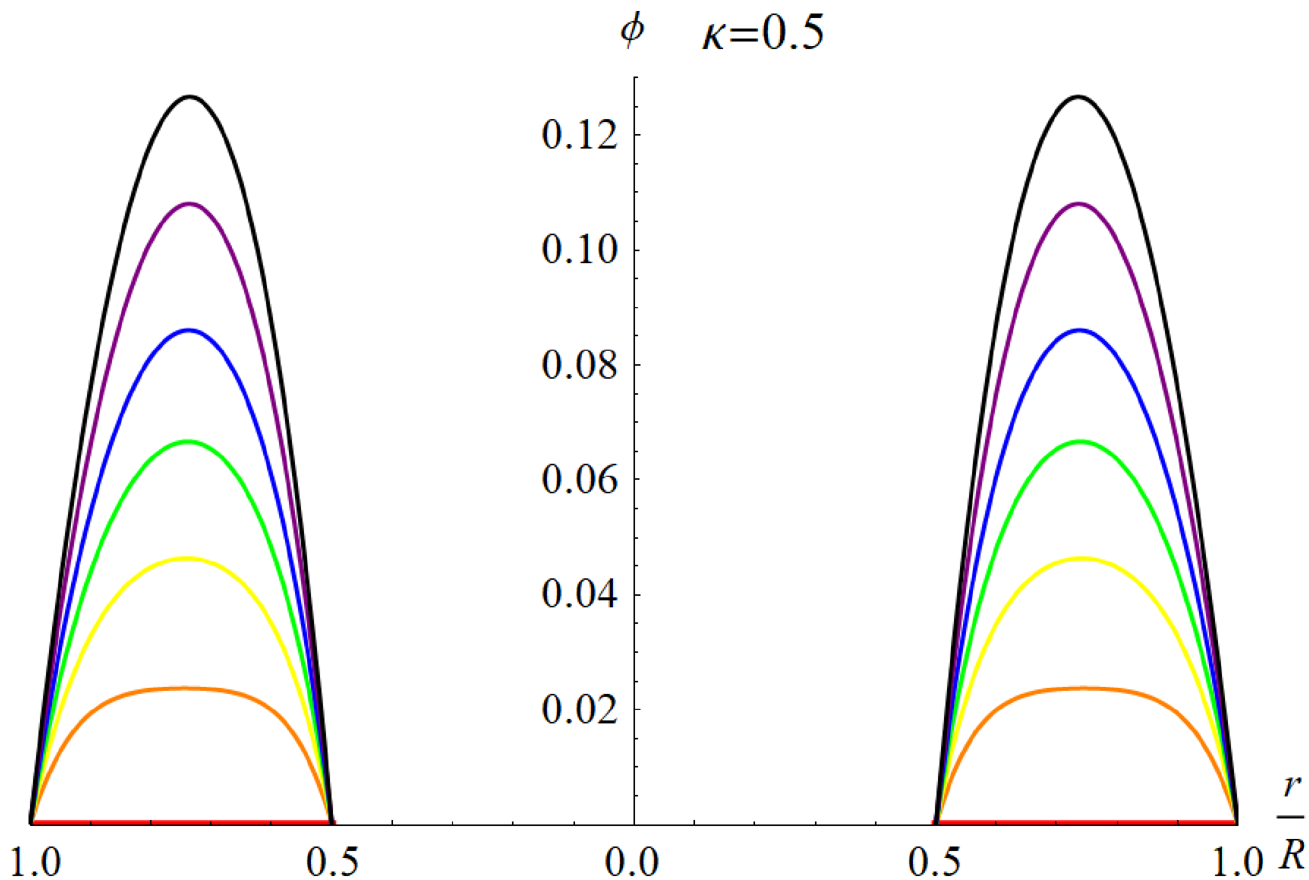

Figure 1: This figure shows the dimensionless velocity distribution ϕ versus ξ for $\kappa = 0.5$ when $\tau = 0$, $\tau = 0.006$, $\tau = 0.0125$, $\tau = 0.02$, $\tau = 0.03$, and $\tau = 0.05$ in red, orange, yellow, green, blue, and purple, respectively. In black is the steady-state velocity distribution. The profiles are only approximate, as only the first 20 terms in the infinite series have been used. The integrals in A_n and the values of γ were calculated numerically. Notice that the maximum velocity occurs at a smaller and smaller radius as time goes on.

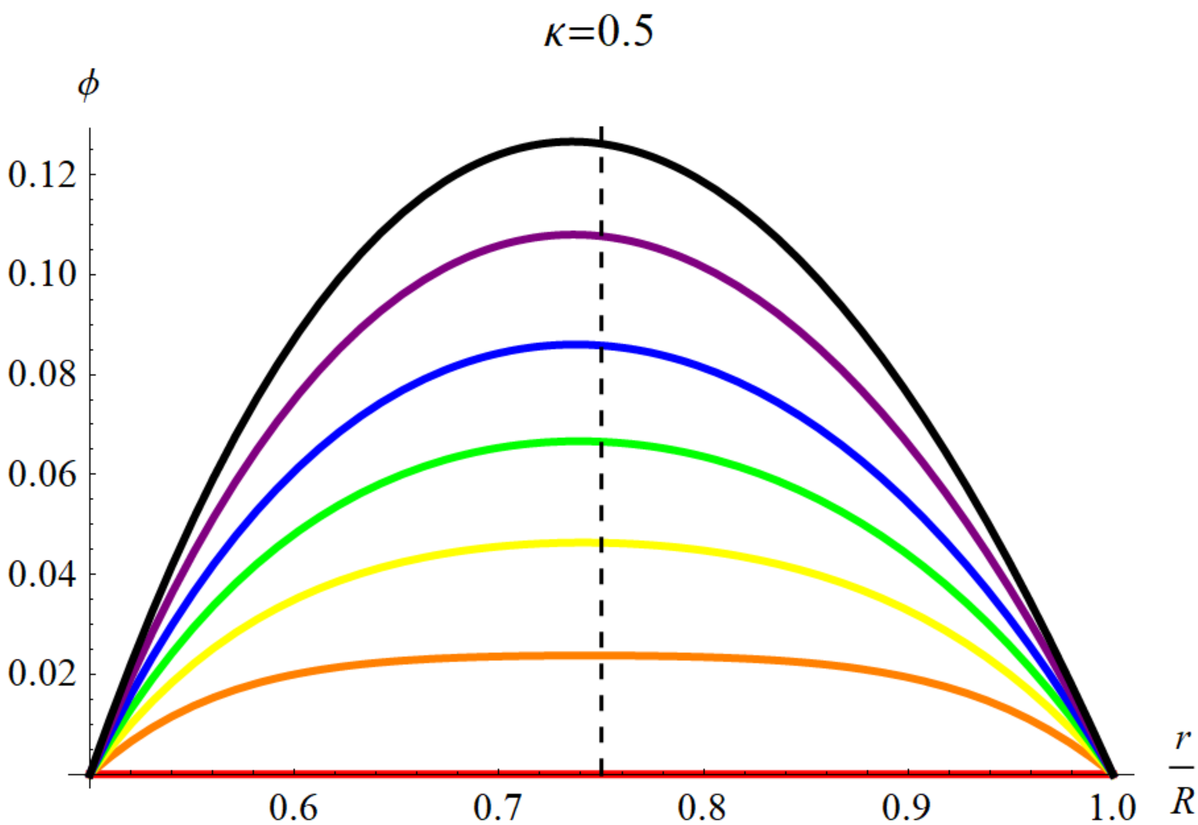

Figure 2: This figure shows the right side of Figure 1 zoomed in. The dashed line marks the center of the annular slit.

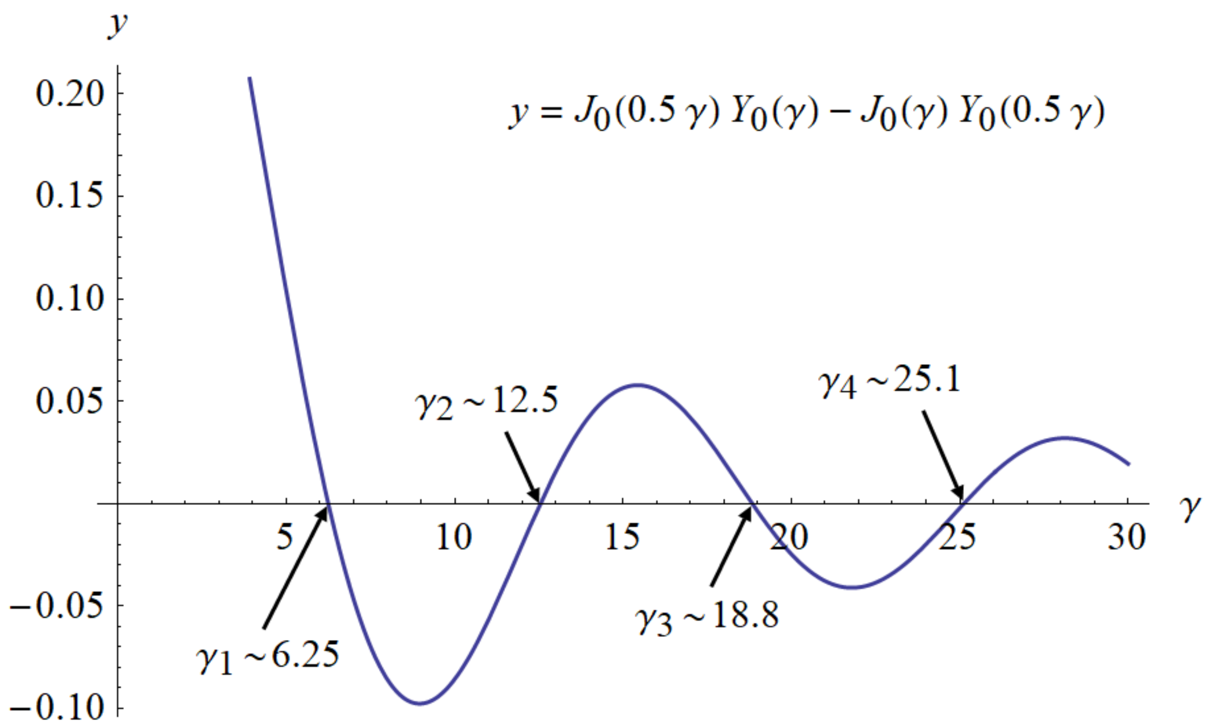

Figure 3: This figure shows a graph of $y = J_0(\gamma\kappa)Y_0(\gamma) - J_0(\gamma)Y_0(\gamma\kappa)$ versus γ for $\kappa = 0.5$.

Part (b)

Since the flow is unsteady, tangential, and independent of height, we assume that the velocity varies as a function of radius and time and that the fluid moves only in the θ-direction.

$$\mathbf{v} = v_\theta(r,t)\hat{\boldsymbol{\theta}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at the inner and outer radii, $r = \kappa R$ and $r = R$, respectively. Let the angular velocity vector be denoted as $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$. The boundary conditions are then

$$\begin{aligned} &\text{Boundary Condition 1:} \quad && v_\theta(\kappa R, t) = 0 \\ &\text{Boundary Condition 2:} \quad && v_\theta(R, t) = \Omega R. \end{aligned}$$

The fluid starts from rest, so the velocity is zero initially.

$$\text{Initial Condition:} \quad v_\theta(r, 0) = 0$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using cylindrical coordinates is the appropriate choice for this problem, so the two previous equations will be used in (r, θ, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_r}_{=0}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\right) = \underbrace{-\frac{1}{r}\frac{\partial p}{\partial \theta}}_{=0} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_\theta}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

The relevant equation for the velocity is the θ-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_\theta(r,t)\hat{\boldsymbol{\theta}}$.

$$\rho\frac{\partial v_\theta}{\partial t} = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)$$

The PDE and its associated initial and boundary conditions will now be nondimensionalized. Divide both sides by μ and introduce R^3 on the right side.

$$\frac{\rho}{\mu}\frac{\partial v_\theta}{\partial t} = R\frac{\partial}{\partial r}\left[\frac{R}{r}R\frac{\partial}{\partial r}\left(\frac{r}{R}v_\theta\right)\right]\frac{1}{R^2}$$

Multiply both sides by R^2.

$$\frac{\rho R^2}{\mu}\frac{\partial v_\theta}{\partial t} = R\frac{\partial}{\partial r}\left[\frac{R}{r}R\frac{\partial}{\partial r}\left(\frac{r}{R}v_\theta\right)\right]$$

Let ξ and τ be defined as in part (a).

$$\frac{\partial v_\theta}{\partial \tau} = \frac{\partial}{\partial \xi}\left(\frac{1}{\xi}\frac{\partial}{\partial \xi}(\xi v_\theta)\right)$$

Divide both sides by ΩR.

$$\frac{\partial}{\partial \tau}\frac{v_\theta}{\Omega R} = \frac{\partial}{\partial \xi}\left[\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{v_\theta}{\Omega R}\right)\right]$$

Introduce a new nondimensional velocity ψ.

$$\psi = \frac{v_\theta}{\Omega R}$$

Then the nondimensional PDE is

$$\frac{\partial \psi}{\partial \tau} = \frac{\partial}{\partial \xi}\left(\frac{1}{\xi}\frac{\partial}{\partial \xi}(\xi\psi)\right)$$

and its initial and boundary conditions are

$$\psi(\kappa,\tau) = 0 \tag{7}$$
$$\psi(1,\tau) = 1 \tag{8}$$
$$\psi(\xi,0) = 0. \tag{9}$$

As a result of the spinning outer cylinder, the fluid starts to move and eventually reaches a steady state. ψ can be thought to have an equilibrium part $\psi_\infty(\xi)$ and a transient part $\psi_t(\xi,\tau)$ that dies out as τ increases. Substitute $\psi(\xi,\tau) = \psi_\infty(\xi) - \psi_t(\xi,\tau)$ into the PDE.

$$\frac{\partial}{\partial \tau}[\psi_\infty(\xi) - \psi_t(\xi,\tau)] = \frac{\partial}{\partial \xi}\left(\frac{1}{\xi}\frac{\partial}{\partial \xi}(\xi[\psi_\infty(\xi) - \psi_t(\xi,\tau)])\right)$$

Distribute the operators.

$$-\frac{\partial \psi_t}{\partial \tau} = \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}(\xi\psi_\infty)\right) - \frac{\partial}{\partial \xi}\left(\frac{1}{\xi}\frac{\partial}{\partial \xi}(\xi\psi_t)\right)$$

If we set

$$\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}(\xi\psi_\infty)\right) = 0, \tag{10}$$

then the previous equation becomes

$$-\frac{\partial \psi_t}{\partial \tau} = -\frac{\partial}{\partial \xi}\left(\frac{1}{\xi}\frac{\partial}{\partial \xi}(\xi \psi_t)\right). \tag{11}$$

Solve equation (10) for the steady-state velocity profile first. Integrate both sides of it with respect to ξ.

$$\frac{1}{\xi}\frac{d}{d\xi}(\xi \psi_\infty) = C_{10}$$

Multiply both sides by ξ.

$$\frac{d}{d\xi}(\xi \psi_\infty) = C_{10}\xi$$

Integrate both sides with respect to ξ once more.

$$\xi \psi_\infty = C_{10}\frac{\xi^2}{2} + C_{11}$$

Divide both sides by ξ.

$$\psi_\infty(\xi) = C_{10}\frac{\xi}{2} + \frac{C_{11}}{\xi}$$

The boundary conditions, $\psi(\kappa, \tau) = 0$ and $\psi(1, \tau) = 1$, hold for all time, including the steady state. Apply $\psi_\infty(\kappa) = 0$ and $\psi_\infty(1) = 1$ to determine C_{10} and C_{11}.

$$\begin{aligned} \psi_\infty(\kappa) &= C_{10}\frac{\kappa}{2} + \frac{C_{11}}{\kappa} = 0 \\ \psi_\infty(1) &= C_{10}\frac{1}{2} + C_{11} = 1 \end{aligned}$$

Solving this system of equations gives

$$C_{10} = \frac{2}{1-\kappa^2} \quad \text{and} \quad C_{11} = -\frac{\kappa^2}{1-\kappa^2}.$$

So the steady-state velocity distribution is

$$\begin{aligned} \psi_\infty(\xi) &= \frac{2}{1-\kappa^2}\frac{\xi}{2} - \frac{\kappa^2}{1-\kappa^2}\frac{1}{\xi} \\ &= \frac{\xi}{1-\kappa^2}\left(1 - \frac{\kappa^2}{\xi^2}\right) \\ &= \frac{\xi}{1-\kappa^2}\left[1 - \left(\frac{\kappa}{\xi}\right)^2\right]. \end{aligned}$$

Now we will solve equation (11) for ψ_t. Multiply both sides of it by -1.

$$\frac{\partial \psi_t}{\partial \tau} = \frac{\partial}{\partial \xi}\left(\frac{1}{\xi}\frac{\partial}{\partial \xi}(\xi \psi_t)\right). \tag{11}$$

To find the initial and boundary conditions associated with it, substitute $\psi(\xi, \tau) = \psi_\infty(\xi) - \psi_t(\xi, \tau)$ into equations (7), (8), and (9).

$$\begin{aligned} \psi(\kappa, \tau) &= \psi_\infty(\kappa) - \psi_t(\kappa, \tau) = 0 - \psi_t(\kappa, \tau) = 0 & \rightarrow \quad & \psi_t(\kappa, \tau) = 0 \\ \psi(1, \tau) &= \psi_\infty(1) - \psi_t(1, \tau) = 1 - \psi_t(1, \tau) = 1 & \rightarrow \quad & \psi_t(1, \tau) = 0 \\ \psi(\xi, 0) &= \psi_\infty(\xi) - \psi_t(\xi, 0) = 0 & \rightarrow \quad & \psi_t(\xi, 0) = \psi_\infty(\xi) \end{aligned}$$

The PDE for ψ_t and its boundary conditions are linear and homogeneous, so the problem can be solved with the method of separation of variables. Assume a product solution of the form $\psi_t = F(\xi)G(\tau)$ and plug it into the PDE

$$FG' = \frac{\partial}{\partial \xi}\left(\frac{1}{\xi}\frac{\partial}{\partial \xi}(\xi F G)\right)$$

and the boundary conditions.

$$\begin{aligned} \psi_t(\kappa, \tau) = 0 \quad &\rightarrow \quad F(\kappa)G(\tau) = 0 \quad \rightarrow \quad F(\kappa) = 0 \\ \psi_t(1, \tau) = 0 \quad &\rightarrow \quad F(1)G(\tau) = 0 \quad \rightarrow \quad F(1) = 0 \end{aligned}$$

Now separate variables in the PDE: bring the constants and functions of τ to the left side and bring the functions of ξ to the right side.

$$\frac{G'}{G} = \frac{1}{F}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}(\xi F)\right)$$

The only way a function of τ can be equal to a function of ξ is if both are equal to a constant η.

$$\frac{G'}{G} = \frac{1}{F}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}(\xi F)\right) = \eta$$

Values of η for which the boundary conditions are satisfied are known as the eigenvalues, and the nontrivial functions associated with them are known as the eigenfunctions.

Determination of Positive Eigenvalues: $\eta = \alpha^2$

Assuming η is positive, the differential equation for G becomes

$$\frac{G'}{G} = \alpha^2.$$

Multiply both sides by G.

$$G' = \alpha^2 G$$

The general solution is written in terms of the exponential function.

$$G(\tau) = C_{12}e^{\alpha^2 t}$$

The possibility that there are positive eigenvalues can be dismissed here because $G(\tau)$ diverges as $\tau \rightarrow \infty$.

Determination of the Zero Eigenvalue: $\eta = 0$

Assuming η is zero, the differential equation for F becomes

$$\frac{1}{F}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}(\xi F)\right) = 0.$$

Multiply both sides by F.

$$\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}(\xi F)\right) = 0$$

Integrate both sides with respect to ξ.

$$\frac{1}{\xi}\frac{d}{d\xi}(\xi F) = C_{13}$$

Multiply both sides by ξ.

$$\frac{d}{d\xi}(\xi F) = C_{13}\xi$$

Integrate both sides with respect to ξ once more.

$$\xi F = C_{13}\frac{\xi^2}{2} + C_{14}$$

Divide both sides by r.

$$F(\xi) = C_{13}\frac{\xi}{2} + \frac{C_{14}}{\xi}$$

Apply the boundary conditions here to determine C_{13} and C_{14}.

$$F(\kappa) = C_{13}\frac{\kappa}{2} + \frac{C_{14}}{\kappa} = 0$$
$$F(1) = C_{13}\frac{1}{2} + C_{14} = 0$$

Solving the system yields $C_{13} = 0$ and $C_{14} = 0$, which results in the trivial solution. Zero is not eigenvalue.

Determination of Negative Eigenvalues: $\eta = -\beta^2$

Assuming η is negative, the differential equation for F becomes

$$\frac{1}{F}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}(\xi F)\right) = -\beta^2.$$

Multiply both sides by $\xi^2 F$.

$$\xi^2\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}(\xi F)\right) = -\beta^2\xi^2 F$$

Expand the left side.

$$\xi^2 F'' + \xi F' - F = -\beta^2\xi^2 F$$

Bring $\beta^2\xi^2 F$ to the left side.

$$\xi^2 F'' + \xi F' + (\beta^2\xi^2 - 1)F = 0$$

This is the parametric form of Bessel's equation of order one. The general solution is written in terms of first-order Bessel functions of the first kind $J_1(\beta\xi)$ and second kind $Y_1(\beta\xi)$.

$$F(\xi) = C_{15}J_1(\beta\xi) + C_{16}Y_1(\beta\xi)$$

Apply the boundary conditions here to determine C_{15} and C_{16}.

$$F(\kappa) = C_{15}J_1(\beta\kappa) + C_{16}Y_1(\beta\kappa) = 0$$
$$F(1) = C_{15}J_1(\beta) + C_{16}Y_1(\beta) = 0$$

Solve the second equation for C_{16}

$$C_{16} = -C_{15}\frac{J_1(\beta)}{Y_1(\beta)}$$

and plug the result into the first equation.

$$C_{15}J_1(\beta\kappa) - C_{15}\frac{J_1(\beta)}{Y_1(\beta)}Y_1(\beta\kappa) = 0$$

To avoid getting the trivial solution, we insist that $C_{15} \neq 0$. Write the two terms on the left side as one.

$$C_{15}\frac{J_1(\beta\kappa)Y_1(\beta) - J_1(\beta)Y_1(\beta\kappa)}{Y_1(\beta)} = 0$$

Hence, β is defined implicitly by $J_1(\beta\kappa)Y_1(\beta) - J_1(\beta)Y_1(\beta\kappa) = 0$. J_1 and Y_1 are oscillatory functions, so there are infinitely many values of β. If β_n denotes the nth zero of the function, then

$$\boxed{J_1(\beta_n\kappa)Y_1(\beta_n) - J_1(\beta_n)Y_1(\beta_n\kappa) = 0, \quad n = 1, 2, \ldots.}$$

The eigenfunctions associated with these eigenvalues for λ are

$$\begin{aligned}
F(\xi) &= C_{15}J_1(\beta\xi) + C_{16}Y_1(\beta\xi) \\
&= C_{15}J_1(\beta\xi) - C_{15}\frac{J_1(\beta)}{Y_1(\beta)}Y_1(\beta\xi) \\
&= \frac{C_{15}}{Y_1(\beta)}[J_1(\beta\xi)Y_1(\beta) - J_1(\beta)Y_1(\beta\xi)] \\
&= C_{17}[J_1(\beta\xi)Y_1(\beta) - J_1(\beta)Y_1(\beta\xi)] \\
&\qquad \rightarrow \quad \boxed{F_n(\xi) = J_1(\beta_n\xi)Y_1(\beta_n) - J_1(\beta_n)Y_1(\beta_n\xi), \quad n = 1, 2, \ldots.}
\end{aligned}$$

Now the ODE for G will be solved.

$$\frac{G'}{G} = -\beta^2$$

Multiply both sides by G.

$$G' = -\beta^2 G$$

The general solution is written in terms of the exponential function.

$$G(\tau) = e^{-\beta^2\tau} \quad \rightarrow \quad G_n(\tau) = e^{-\beta_n^2\tau}, \quad n = 1, 2, \ldots$$

According to the principle of superposition, the general solution for ψ_t is a linear combination of eigenfunctions over all the eigenvalues.

$$\psi_t(\xi, \tau) = \sum_{n=1}^{\infty} B_n e^{-\beta_n^2\tau} F_n(\xi)$$

Apply the initial condition here to determine B_n.

$$\psi_t(\xi, 0) = \sum_{n=1}^{\infty} B_n F_n(\xi) = \psi_\infty(\xi)$$

Substitute the steady-state velocity distribution found for ψ_∞.

$$\sum_{n=1}^{\infty} B_n F_n(\xi) = \frac{\xi}{1-\kappa^2}\left[1-\left(\frac{\kappa}{\xi}\right)^2\right]$$

Multiply both sides by $F_m(\xi)\xi$, where m is an integer.

$$\sum_{n=1}^{\infty} B_n F_n(\xi) F_m(\xi)\xi = \frac{\xi}{1-\kappa^2}\left[1-\left(\frac{\kappa}{\xi}\right)^2\right] F_m(\xi)\xi$$

Integrate both sides with respect to ξ from κ to 1 and distribute ξ^2.

$$\int_\kappa^1 \sum_{n=1}^{\infty} B_n F_n(\xi) F_m(\xi)\xi \, d\xi = \int_\kappa^1 \frac{\xi^2-\kappa^2}{1-\kappa^2} F_m(\xi)\, d\xi$$

Bring the constants in front of the integral on the left side.

$$\sum_{n=1}^{\infty} B_n \int_\kappa^1 F_n(\xi) F_m(\xi)\xi \, d\xi = \int_\kappa^1 \frac{\xi^2-\kappa^2}{1-\kappa^2} F_m(\xi)\, d\xi$$

Because the $F_n(\xi)$ satisfy an ODE of the Sturm-Liouville form, they are guaranteed to be orthogonal with respect to the weight ξ, meaning that the integral on the left side is zero for $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$B_n \int_\kappa^1 F_n^2(\xi)\xi \, d\xi = \int_\kappa^1 \frac{\xi^2-\kappa^2}{1-\kappa^2} F_n(\xi)\, d\xi$$

Solve this equation for B_n.

$$\boxed{B_n = \frac{\displaystyle\int_\kappa^1 \frac{\xi^2-\kappa^2}{1-\kappa^2} F_n(\xi)\, d\xi}{\displaystyle\int_\kappa^1 F_n^2(\xi)\xi \, d\xi}}$$

The dimensionless velocity is then

$$\psi(\xi,\tau) = \frac{\xi}{1-\kappa^2}\left[1-\left(\frac{\kappa}{\xi}\right)^2\right] - \sum_{n=1}^{\infty} B_n e^{-\beta_n^2 \tau} F_n(\xi).$$

Changing back to the original variables, the velocity distribution for $t > 0$ in an annulus with the outer radius rotating at angular velocity $\Omega\hat{\mathbf{z}}$ is therefore

$$\boxed{v_\theta(r,t) = \frac{\Omega r}{1-\kappa^2}\left[1-\left(\frac{\kappa R}{r}\right)^2\right] - \sum_{n=1}^{\infty} B_n \exp\left(-\beta_n^2 \frac{\mu}{\rho R^2} t\right) F_n\left(\frac{r}{R}\right).}$$

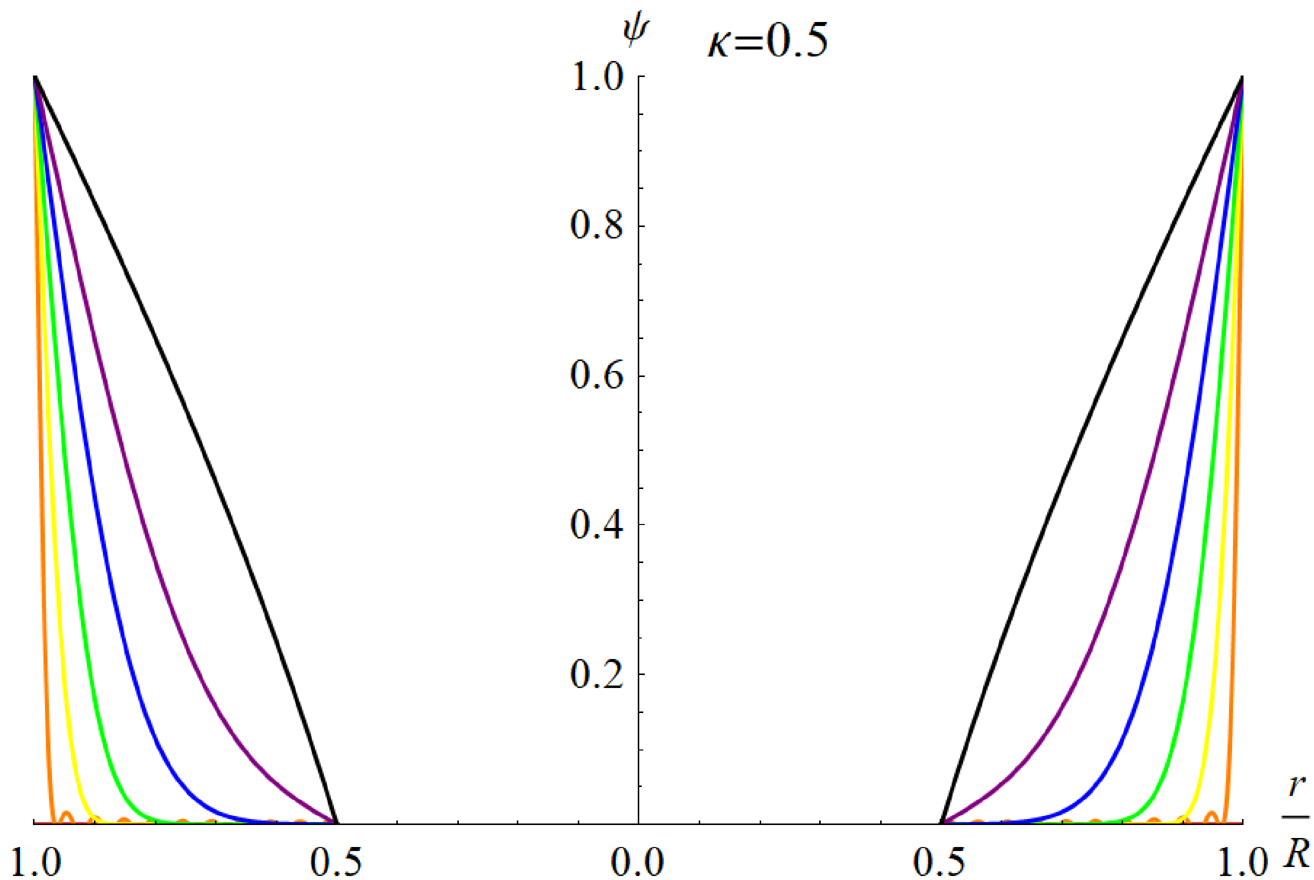

Figure 4: This figure shows the dimensionless velocity distribution ψ versus ξ for $\kappa = 0.5$ when $\tau = 0$, $\tau = 0.0001$, $\tau = 0.0007$, $\tau = 0.0025$, $\tau = 0.0075$, and $\tau = 0.02$ in red, orange, yellow, green, blue, and purple, respectively. In black is the steady-state velocity distribution. The profiles are only approximate, as only the first 20 terms in the infinite series have been used. The integrals in B_n and the values of β were calculated numerically. Notice that the steady-state velocity is not linear.

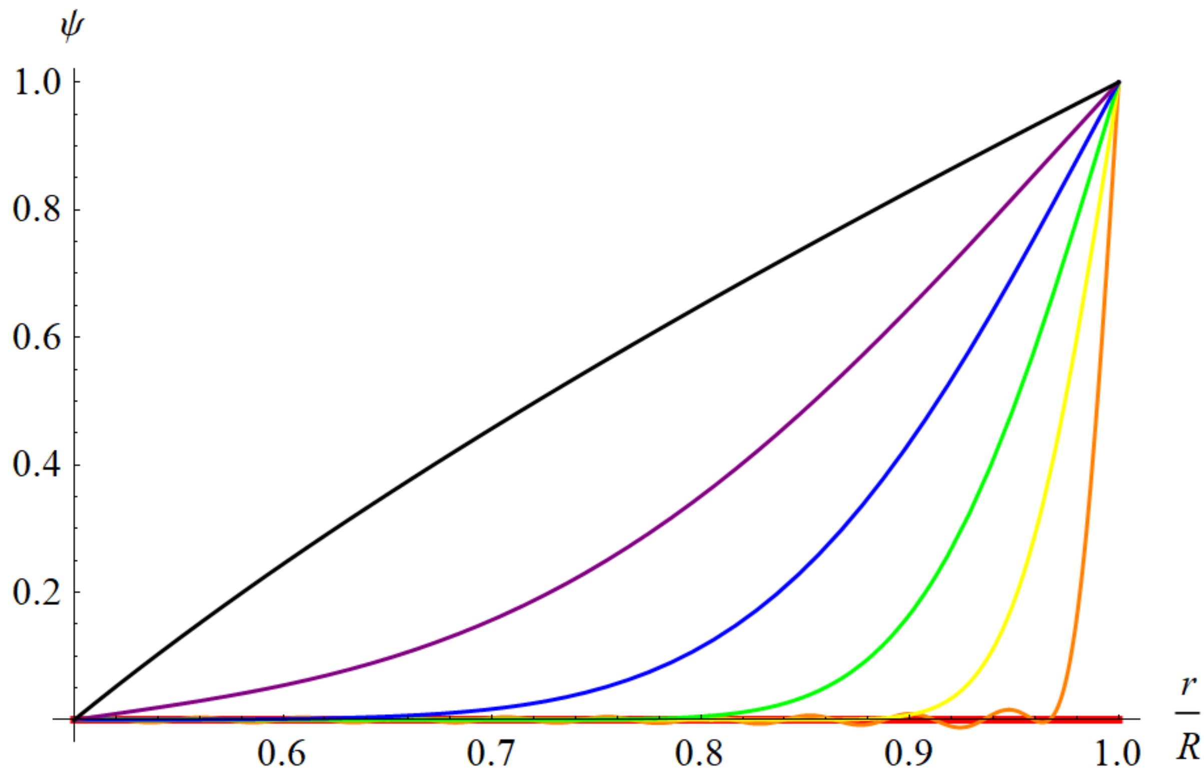

Figure 5: This figure shows the right side of Figure 4 zoomed in.

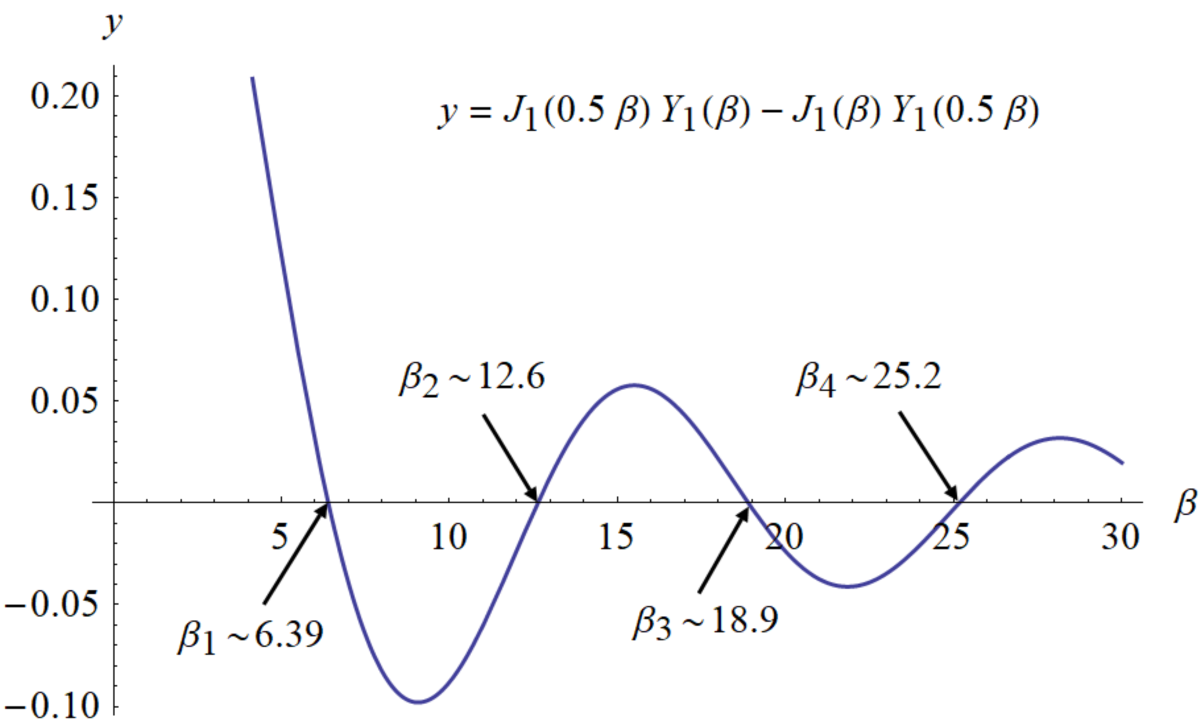

Figure 6: This figure shows a graph of $y = J_1(\beta\kappa)Y_1(\beta) - J_1(\beta)Y_1(\beta\kappa)$ versus β for $\kappa = 0.5$.