Problem 4D.5

Stream functions for three-dimensional flow.

(a) Show that the velocity functions $\rho\mathbf{v} = [\nabla \times \mathbf{A}]$ and $\rho\mathbf{v} = [(\nabla\psi_1) \times (\nabla\psi_2)]$ both satisfy the equation of continuity identically for steady flow. The second function also describes unsteady incompressible flows. The functions ψ_1, ψ_2, and $\mathbf{A}$ are arbitrary, except that their derivatives appearing in $(\nabla \cdot \rho\mathbf{v})$ must exist.

(b) Show that the expression $\mathbf{A}/\rho = -\boldsymbol{\delta}_3\psi/h_3$ reproduces the velocity components for the four incompressible flows of Table 4.2-1. Here h_3 and $\boldsymbol{\delta}_3$ are the scale factor and unit vector for the velocity component not shown in the table. (Read the general vector $\mathbf{v}$ of Eq. A.7-18 here as $\mathbf{A}$.)

(c) Show that the streamlines of $[(\nabla\psi_1) \times (\nabla\psi_2)]$ are given by the intersections of the surfaces $\psi_1 = \text{constant}$ and $\psi_2 = \text{constant}$. Sketch such a pair of surfaces for the flow in Fig. 4.3-1.

(d) Use Stokes' theorem (Eq. A.5-4), and the definition of $\mathbf{A}$ from (a), to obtain an expression in terms of $\mathbf{A}$ for the mass flow rate through a surface S bounded by a closed curve C. Show that the vanishing of $\mathbf{v}$ on C does not imply the vanishing of $\mathbf{A}$ on C.

Solution

Part (a)

The equation of continuity is

$$\frac{\partial\rho}{\partial t} = -(\nabla \cdot \rho\mathbf{v}).$$

The left side is equal to zero for steady flow. Consequently, we have to show that the given velocity functions satisfy

$$\nabla \cdot \rho\mathbf{v} = 0.$$

For the first velocity function $\rho\mathbf{v} = [\nabla \times \mathbf{A}]$, we have

$$\nabla \cdot \rho\mathbf{v} = \sum_{k=1}^{3} \frac{\partial}{\partial x_k}[\nabla \times \mathbf{A}]_k = \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ijk} \frac{\partial}{\partial x_i} A_j = \sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ijk} \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i} A_j.$$

Switch the dummy indices: replace i with k and k with i.

$$= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{j=1}^{3} \varepsilon_{kji} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} A_j$$

Since the limits of summation are constant, the sums can be ordered however we like. The mixed partial derivatives are equal by Clairaut's theorem, so they can also be ordered however we like.

$$= \sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{kji} \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i} A_j$$

Permute the indices of the Levi-Civita symbol: $\varepsilon_{kji} = -\varepsilon_{ijk}$.

$$= -\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ijk} \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i} A_j = 0$$

For the second velocity function $\rho\mathbf{v} = [(\nabla\psi_1)\times(\nabla\psi_2)]$, we have

$$\begin{aligned}\nabla\cdot\rho\mathbf{v} = \sum_{k=1}^{3}\frac{\partial}{\partial x_k}[(\nabla\psi_1)\times(\nabla\psi_2)]_k &= \sum_{k=1}^{3}\frac{\partial}{\partial x_k}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{ijk}(\nabla\psi_1)_i(\nabla\psi_2)_j \\ &= \sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{ijk}\frac{\partial}{\partial x_k}\frac{\partial\psi_1}{\partial x_i}\frac{\partial\psi_2}{\partial x_j}.\end{aligned}$$

Switch the dummy indices: replace i with k and k with i.

$$= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{j=1}^{3}\varepsilon_{kji}\frac{\partial}{\partial x_i}\frac{\partial\psi_1}{\partial x_k}\frac{\partial\psi_2}{\partial x_j}$$

Since the limits of summation are constant, the sums can be ordered however we like. The mixed partial derivatives are equal by Clairaut's theorem, so they can also be ordered however we like.

$$= \sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{kji}\frac{\partial}{\partial x_k}\frac{\partial\psi_1}{\partial x_i}\frac{\partial\psi_2}{\partial x_j}$$

Permute the indices of the Levi-Civita symbol: $\varepsilon_{kji} = -\varepsilon_{ijk}$.

$$= -\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{ijk}\frac{\partial}{\partial x_k}\frac{\partial\psi_1}{\partial x_i}\frac{\partial\psi_2}{\partial x_j} = 0$$

Part (b)

Substitute the expression for $\mathbf{A}$ into the equation relating it with the velocity $\mathbf{v}$.

$$\begin{aligned}\rho\mathbf{v} &= [\nabla\times\mathbf{A}] \\ &= \nabla\times\left[-\frac{\rho\psi}{h_3}\boldsymbol{\delta}_3\right] \\ &= -\rho\left[\nabla\times\frac{\psi}{h_3}\boldsymbol{\delta}_3\right]\end{aligned}$$

Divide both sides by ρ.

$$\mathbf{v} = -\left[\nabla\times\frac{\psi}{h_3}\boldsymbol{\delta}_3\right]$$

In the first incompressible flow, the coordinate system is rectangular with $v_z = 0$ (no z-dependence). z is the third coordinate in the system here, and since we're in Cartesian coordinates, $h_3 = 1$.

$$\mathbf{v} = -\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi\end{vmatrix} = -\left[\hat{\mathbf{x}}\frac{\partial\psi}{\partial y} - \hat{\mathbf{y}}\frac{\partial\psi}{\partial x}\right] = \hat{\mathbf{x}}\left(-\frac{\partial\psi}{\partial y}\right) + \hat{\mathbf{y}}\left(\frac{\partial\psi}{\partial x}\right)$$

Therefore,

$$\boxed{v_x = -\frac{\partial\psi}{\partial y} \quad \text{and} \quad v_y = \frac{\partial\psi}{\partial x}.}$$

In the second incompressible flow, the coordinate system is cylindrical with $v_z = 0$ (no z-dependence). z is the third coordinate in the system here, and since we're in cylindrical coordinates, $h_3 = 1$.

$$\mathbf{v} = -\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix} = -\left[\hat{\mathbf{r}}\frac{1}{r}\frac{\partial \psi}{\partial \theta} - \hat{\boldsymbol{\theta}}\frac{\partial \psi}{\partial r}\right] = \hat{\mathbf{r}}\left(-\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) + \hat{\boldsymbol{\theta}}\left(\frac{\partial \psi}{\partial r}\right)$$

Therefore,

$$\boxed{v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = \frac{\partial \psi}{\partial r}.}$$

In the third incompressible flow, the coordinate system is cylindrical with $v_\theta = 0$ (no θ-dependence). θ is the third coordinate in the system here, and since we're in cylindrical coordinates, $h_3 = r$.

$$\mathbf{v} = -\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & \frac{\psi}{r} & 0 \end{vmatrix} = -\frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & \psi & 0 \end{vmatrix} = -\frac{1}{r}\left[\hat{\mathbf{r}}\left(-\frac{\partial \psi}{\partial z}\right) + \hat{\mathbf{z}}\frac{\partial \psi}{\partial r}\right] = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial \psi}{\partial z}\right) + \hat{\mathbf{z}}\left(-\frac{1}{r}\frac{\partial \psi}{\partial r}\right)$$

Therefore,

$$\boxed{v_r = \frac{1}{r}\frac{\partial \psi}{\partial z} \quad \text{and} \quad v_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}.}$$

In the fourth incompressible flow, the coordinate system is spherical with $v_\phi = 0$ (no ϕ-dependence). ϕ is the third coordinate in the system here, and since we're in spherical coordinates, $h_3 = r\sin\theta$, where θ represents the angle from the polar axis.

$$\mathbf{v} = -\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\ 0 & 0 & \frac{\psi}{r\sin\theta} \end{vmatrix} = -\frac{1}{r\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \psi \end{vmatrix} = -\frac{1}{r\sin\theta}\left[\hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) - \hat{\boldsymbol{\theta}}\frac{\partial \psi}{\partial r}\right]$$

$$= \hat{\mathbf{r}}\left(-\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta}\right) + \hat{\boldsymbol{\theta}}\left(\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}\right)$$

Therefore,

$$\boxed{v_r = -\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}.}$$

Part (c)

Part (d)

The volumetric flow rate is given by the integral of the velocity field over the surface the fluid is flowing through.

$$\frac{dV}{dt} = \iint_S \mathbf{v} \cdot d\mathbf{S}$$

To get the mass flow rate through the surface, multiply both sides by the density ρ.

$$\rho\frac{dV}{dt} = \rho\iint_S \mathbf{v} \cdot d\mathbf{S}$$

Bring ρ inside the operator on each side.

$$\frac{d(\rho V)}{dt} = \iint_S \rho \mathbf{v} \cdot d\mathbf{S}$$

Density times volume is the mass. Use $\rho \mathbf{v} = \nabla \times \mathbf{A}$ on the right side.

$$\frac{dm}{dt} = \iint_S \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

Since we have a surface integral of a curl, Stokes's theorem can be applied to change it to a closed loop integral over the surface's bounding curve C. Therefore,

$$\boxed{\frac{dm}{dt} = \oint_C \mathbf{A} \cdot d\mathbf{s}.}$$

If $\mathbf{v} = \mathbf{0}$ on C, then $\nabla \times \mathbf{A} = \mathbf{0}$ on C, which means that $\mathbf{A}$ can be written as the gradient of a scalar function: $\mathbf{A} = \nabla f$. Therefore, the vanishing of $\mathbf{v}$ on C implies that $\mathbf{A}$ is irrotational on C, not that $\mathbf{A}$ vanishes on C.