Problem 4D.5

Stream functions for three-dimensional flow.

(a) Show that the velocity functions $\rho\mathbf{v} = [\nabla \times \mathbf{A}]$ and $\rho\mathbf{v} = [(\nabla\psi_1) \times (\nabla\psi_2)]$ both satisfy the equation of continuity identically for steady flow. The second function also describes unsteady incompressible flows. The functions ψ_1, ψ_2, and $\mathbf{A}$ are arbitrary, except that their derivatives appearing in $(\nabla \cdot \rho\mathbf{v})$ must exist.

(b) Show that the expression $\mathbf{A}/\rho = -\boldsymbol{\delta}_3\psi/h_3$ reproduces the velocity components for the four incompressible flows of Table 4.2-1. Here h_3 and $\boldsymbol{\delta}_3$ are the scale factor and unit vector for the velocity component not shown in the table. (Read the general vector $\mathbf{v}$ of Eq. A.7-18 here as $\mathbf{A}$.)

(c) Show that the streamlines of $[(\nabla\psi_1) \times (\nabla\psi_2)]$ are given by the intersections of the surfaces $\psi_1 =$ constant and $\psi_2 =$ constant. Sketch such a pair of surfaces for the flow in Fig. 4.3-1.

(d) Use Stokes' theorem (Eq. A.5-4), and the definition of $\mathbf{A}$ from (a), to obtain an expression in terms of $\mathbf{A}$ for the mass flow rate through a surface S bounded by a closed curve C. Show that the vanishing of $\mathbf{v}$ on C does not imply the vanishing of $\mathbf{A}$ on C.