Problem 1.13

(a) Show that if a is a constant and $b(x)$ is a function, then

$$y'' + \frac{b'(x)}{b(x)}y' - \frac{a^2}{[b(x)]^2}y = 0$$

has a pair of linearly independent solutions which are reciprocals; find them.

(b) $y(x)$ and $[y(x)]^2$ are both solutions of $y'' + p(x)y' + 2y = 0$. Find $y(x)$.

Solution

Part (a)

If y and y^{-1} are both solutions, then that means they both satisfy the ODE. We thus have two equations to work with.

$$\begin{aligned} y'' + \frac{b'(x)}{b(x)}y' - \frac{a^2}{[b(x)]^2}y &= 0 \\ (y^{-1})'' + \frac{b'(x)}{b(x)}(y^{-1})' - \frac{a^2}{[b(x)]^2}y^{-1} &= 0 \end{aligned}$$

Solve the first equation for y'' and evaluate the derivatives in the second equation.

$$\begin{aligned} y'' &= \frac{a^2}{[b(x)]^2}y - \frac{b'(x)}{b(x)}y' \\ \frac{2(y')^2}{y^3} - \frac{y''}{y^2} + \frac{b'(x)}{b(x)}\left(-\frac{y'}{y^2}\right) - \frac{a^2}{[b(x)]^2}\frac{1}{y} &= 0 \end{aligned}$$

Substitute the expression for y'' into the second equation to get an ODE that is first-order for y.

$$\frac{2(y')^2}{y^3} - \frac{1}{y^2}\left[\frac{a^2}{[b(x)]^2}y - \frac{b'(x)}{b(x)}y'\right] + \frac{b'(x)}{b(x)}\left(-\frac{y'}{y^2}\right) - \frac{a^2}{[b(x)]^2}\frac{1}{y} = 0$$

Expand the left side.

$$\frac{2(y')^2}{y^3} - \frac{a^2}{[b(x)]^2}\frac{1}{y} + \cancel{\frac{b'(x)}{b(x)}\frac{y'}{y^2}} - \cancel{\frac{b'(x)}{b(x)}\frac{y'}{y^2}} - \frac{a^2}{[b(x)]^2}\frac{1}{y} = 0$$

Combine like-terms.

$$\frac{2(y')^2}{y^3} - \frac{2a^2}{[b(x)]^2}\frac{1}{y} = 0$$

Multiply both sides by y^3 and divide both sides by 2.

$$(y')^2 - \frac{a^2}{[b(x)]^2}y^2 = 0$$

The left side is a difference of squares, so it can be factored.

$$\left[\frac{dy}{dx} + \frac{a}{b(x)}y\right]\left[\frac{dy}{dx} - \frac{a}{b(x)}y\right] = 0$$

By the zero product theorem, we have

$$\frac{dy}{dx} + \frac{a}{b(x)}y = 0 \quad \text{or} \quad \frac{dy}{dx} - \frac{a}{b(x)}y.$$

Both of these ODEs for y can be solved with separation of variables.

$$\frac{dy}{dx} = -\frac{a}{b(x)}y \qquad\qquad \frac{dy}{dx} = \frac{a}{b(x)}y$$

Separate variables.

$$\frac{dy}{y} = -\frac{a}{b(x)}\,dx \qquad\qquad \frac{dy}{y} = \frac{a}{b(x)}\,dx$$

Integrate both sides.

$$\ln|y| = -\int^x \frac{a}{b(s)}\,ds + C_1 \qquad\qquad \ln|y| = \int^x \frac{a}{b(s)}\,ds + C_2$$

Exponentiate both sides.

$$|y| = e^{-\int^x \frac{a}{b(s)}\,ds}e^{C_1} \qquad\qquad |y| = e^{\int^x \frac{a}{b(s)}\,ds}e^{C_2}$$

Introduce $\pm$ on the right side to remove the absolute value sign on the left.

$$y(x) = \pm e^{C_1}e^{-\int^x \frac{a}{b(s)}\,ds} \qquad\qquad y(x) = \pm e^{C_2}e^{\int^x \frac{a}{b(s)}\,ds}$$

Use new arbitrary constants.

$$y(x) = Ae^{-\int^x \frac{a}{b(s)}\,ds} \qquad\qquad y(x) = Be^{\int^x \frac{a}{b(s)}\,ds}$$

Therefore, the two linearly independent reciprocal solutions to the ODE are

$$y_1(x) = \frac{1}{e^{\int^x \frac{a}{b(s)}\,ds}} \quad \text{and} \quad y_2(x) = e^{\int^x \frac{a}{b(s)}\,ds}.$$

Part (b)

If $y(x)$ and $[y(x)]^2$ are both solutions, then that means they both satisfy the ODE. We thus have two equations to work with.

$$\begin{aligned} &y'' + p(x)y' + 2y = 0 \\ &(y^2)'' + p(x)(y^2)' + 2y^2 = 0 \end{aligned}$$

Solve the first equation for y'' and evaluate the derivatives in the second equation.

$$\begin{aligned} &y'' = -p(x)y' - 2y \\ &2(y')^2 + 2yy'' + p(x)\cdot 2yy' + 2y^2 = 0 \end{aligned}$$

Substitute the expression for y'' into the second equation to get an ODE that is first-order for y.

$$2(y')^2 + 2y[-p(x)y' - 2y] + p(x)\cdot 2yy' + 2y^2 = 0$$

Expand the left side.

$$2(y')^2 - \cancel{2p(x)yy'} - 4y^2 + \cancel{2p(x)yy'} + 2y^2 = 0$$

Combine like-terms.

$$2(y')^2 - 2y^2 = 0$$

Divide both sides by 2.

$$(y')^2 - y^2 = 0$$

The left side is a difference of squares, so it can be factored.

$$\left(\frac{dy}{dx} + y\right)\left(\frac{dy}{dx} - y\right) = 0$$

We have the following from the zero product theorem.

$$\frac{dy}{dx} + y = 0 \quad \text{or} \quad \frac{dy}{dx} - y = 0$$

Both of these are first-order ODEs we can solve with separation of variables.

$$\frac{dy}{dx} = -y \qquad\qquad \frac{dy}{dx} = y$$

Separate variables.

$$\frac{dy}{y} = -dx \qquad\qquad \frac{dy}{y} = dx$$

Integrate both sides.

$$\ln|y| = -x + C_1 \qquad\qquad \ln|y| = x + C_2$$

Exponentiate both sides.

$$|y| = e^{-x}e^{C_1} \qquad\qquad |y| = e^{x}e^{C_2}$$

Introduce $\pm$ on the right side to remove the absolute value sign on the left.

$$y(x) = \pm e^{C_1}e^{-x} \qquad\qquad y(x) = \pm e^{C_2}e^{x}$$

Therefore, we have

$$y(x) = Ce^{\pm x},$$

where C is an arbitrary constant.

The general solution to the ODE, $y'' + p(x)y' + 2y = 0$, is quite complicated, but if y and y^2 both happen to be solutions, then $p(x)$ has to equal ∓ 3. To demonstrate this point, note that the general solution to $y'' + 3y' + 2y = 0$ is $y(x) = Ae^{-x} + Be^{-2x}$, and the general solution to $y'' - 3y' + 2y = 0$ is $y(x) = Ae^{x} + Be^{2x}$.