Problem 1.28

Discuss the existence and uniqueness of solutions to the initial-value problem $y' = \sqrt{1-y^2}$ $[y(0) = a]$, for all initial values a. Is there a unique solution if $a = 1$?

Solution

The square root function on the right side exists and is continuous for any neighborhood of x and y when $-1 \leq y \leq 1$. However, the derivative of it with respect to y,

$$\frac{\partial}{\partial y}\sqrt{1-y^2} = \frac{1}{2}(1-y^2)^{-1/2}(-2y) = -\frac{y}{\sqrt{1-y^2}},$$

is not continuous when y is in the neighborhood of 1 or -1. The conditions of the existence and uniqueness theorem are not satisfied here, so uniqueness is not guaranteed. Consequently, for the initial-value problem, $y' = \sqrt{1-y^2}$ $[y(0) = 1]$, there may be more than one solution. Indeed, by inspection we see that $y(x) = 1$ satisfies it and by separation of variables a second solution can be obtained.

$$\frac{dy}{\sqrt{1-y^2}} = dx$$

Integrate both sides.

$$\text{arc}\sin y = x + C$$

Take the sine of both sides.

$$y(x) = \sin(x + C)$$

Determine the constant of integration by using the initial condition.

$$y(0) = \sin(C) = 1 \quad \rightarrow \quad C = \frac{\pi}{2}$$

Hence,

$$y(x) = \sin\left(x + \frac{\pi}{2}\right)$$

is a second solution. The discussion is summarized below.

$$\begin{cases} a^2 > 1 & \text{No solution to the IVP exists because the square root is undefined} \\ a^2 = 1 & \text{A solution to the IVP exists but is not unique because the derivative is not continuous} \\ a^2 < 1 & \text{A solution to the IVP exists and is unique} \end{cases}$$