Problem 1.31

Solve the following differential equations:

(c) $y' = x^2 + 2xy + y^2$;

Solution

The right side is a perfect square.

$$y' = (x + y)^2$$

It suggests the substitution,

$$u = x + y \quad \rightarrow \quad u - x = y$$

$$\frac{du}{dx} - 1 = \frac{dy}{dx}$$

Plugging these into the ODE gives us

$$\frac{du}{dx} - 1 = u^2.$$

This equation can be solved by separation of variables.

$$\frac{du}{dx} = u^2 + 1$$

$$\frac{du}{u^2 + 1} = dx$$

Integrate both sides.

$$\text{arc}\tan u = x + C$$

Take the tangent of both sides.

$$u(x) = \tan(x + C)$$

Now change back to the original variable y.

$$x + y = \tan(x + C)$$

Therefore,

$$y(x) = \tan(x + C) - x.$$