Problem 1.31

Solve the following differential equations:

(f) $x^2y' + xy + y^2 = 0;$

Solution

This is a Bernoulli equation, so we start by dividing both sides by y^2.

$$x^2y^{-2}y' + xy^{-1} + 1 = 0$$

Now make the substitution,

$$\begin{aligned} u &= y^{-1} \\ \frac{du}{dx} &= -y^{-2}\frac{dy}{dx} \quad \to \quad -\frac{du}{dx} = y^{-2}\frac{dy}{dx} \end{aligned}$$

Plug these into the ODE.

$$x^2\left(-\frac{du}{dx}\right) + xu + 1 = 0$$

Divide both sides by $-x^2$.

$$\frac{du}{dx} - \frac{1}{x}u - \frac{1}{x^2} = 0$$

Bring $1/x^2$ to the right side.

$$\frac{du}{dx} - \frac{1}{x}u = \frac{1}{x^2}$$

This is a first-order inhomogeneous ODE that can be solved by multiplying both sides by an integrating factor.

$$I = e^{\int^x -\frac{1}{s}\,ds} = e^{-\ln x} = x^{-1}$$

Proceed with the multiplication of both sides by I.

$$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = \frac{1}{x^3}$$

The left side is now exact and can be written as $d/dx(Iu)$ as a result of the product rule.

$$\frac{d}{dx}\left(\frac{1}{x}u\right) = \frac{1}{x^3}$$

Integrate both sides with respect to x.

$$\frac{1}{x}u = -\frac{1}{2x^2} + C$$

Multiply both sides by x to solve for u.

$$u(x) = -\frac{1}{2x} + Cx$$

Now that the integration is done, change back to the original variable y.

$$\frac{1}{y} = -\frac{1}{2x} + Cx$$

Combine the terms on the right side.

$$\frac{1}{y} = \frac{-1 + 2Cx^2}{2x}$$

Invert both sides to solve for y.

$$y(x) = \frac{2x}{2Cx^2 - 1}$$

Introduce a new arbitrary constant, A, to eliminate the 2 in the denominator. Therefore,

$$y(x) = \frac{2x}{Ax^2 - 1}.$$