Problem 1.31

Solve the following differential equations:

(h) $(x+y^2)+2(y^2+y+x-1)y'=0$, using an integrating factor of the form $I(x,y)=e^{ax+by}$;

Solution

This differential equation is of the form,

$$M(x,y)+N(x,y)\frac{dy}{dx}=0.$$

Multiplying both sides by an integrating factor $I(x,y)$ gives

$$I(x,y)M(x,y)+I(x,y)N(x,y)\frac{dy}{dx}=0. \tag{1}$$

Our aim is to determine the constants, a and b, in the provided function so that

$$\frac{\partial}{\partial y}I(x,y)M(x,y)=\frac{\partial}{\partial x}I(x,y)N(x,y).$$

This is the condition that has to hold in order for the ODE to be exact. Using the product rule, we have for the left side

$$\begin{aligned}\frac{\partial}{\partial y}I(x,y)M(x,y)&=\frac{\partial}{\partial y}(x+y^2)e^{ax+by}\\&=2ye^{ax+by}+(x+y^2)be^{ax+by}\\&=[2y+b(x+y^2)]e^{ax+by}.\end{aligned}$$

Using the product rule, we have for the right side

$$\begin{aligned}\frac{\partial}{\partial x}I(x,y)N(x,y)&=\frac{\partial}{\partial x}2(y^2+y+x-1)e^{ax+by}\\&=2e^{ax+by}+2(y^2+y+x-1)ae^{ax+by}\\&=2[1+a(y^2+y+x-1)]e^{ax+by}.\end{aligned}$$

In order for these partial derivatives to be equal, we require that

$$2y+b(x+y^2)=2[1+a(y^2+y+x-1)].$$

Expand both sides of the equation.

$$2y+bx+by^2=2+2ay^2+2ay+2ax-2a$$

This equation can only be true if we set $a=1$ and $b=2$. Thus, our integrating factor is $I(x,y)=e^{x+2y}$. The ODE we started with becomes exact as a result of multiplying both sides of it by this integrating factor. The fact that it is exact means there exists a potential function $\phi(x,y)$ such that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=I(x,y)M(x,y)\\\frac{\partial\phi}{\partial y}&=I(x,y)N(x,y).\end{aligned}$$

The ODE in equation (1) can hence be written as

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0.$$

Recall that for a function of two variables $\phi(x, y)$, its differential is defined as

$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy.$$

Dividing both sides by dx yields the relationship between the total derivative of a function and its partial derivatives.

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

So the ODE reduces to

$$\frac{d\phi}{dx} = 0.$$

Integrating both sides with respect to x gives the general solution.

$$\phi(x, y) = A,$$

where A is an arbitrary constant. Our goal now is to find this potential function.

$$\frac{\partial \phi}{\partial x} = (x + y^2)e^{x+2y} \tag{2}$$

$$\frac{\partial \phi}{\partial y} = 2(y^2 + y + x - 1)e^{x+2y} \tag{3}$$

Since equation (2) looks simpler, integrate both sides of it partially with respect to x to solve for ϕ. Note that we would arrive at the same answer if we integrated both sides of equation (3) partially with respect to y.

$$\begin{aligned}
\phi(x, y) &= \int^x \left.\frac{\partial \phi}{\partial x}\right|_{x=s} ds + f(y) \\
&= \int^x (s + y^2)e^{s+2y}\,ds + f(y) \\
&= \int^x (se^s e^{2y} + y^2 e^s e^{2y})\,ds + f(y) \\
&= e^{2y}\int^x se^s\,ds + y^2 e^{2y}\int^x e^s\,ds + f(y) \\
&= e^{2y}(x - 1)e^x + y^2 e^{2y} e^x + f(y) \\
&= (x - 1 + y^2)e^{x+2y} + f(y),
\end{aligned}$$

where $f(y)$ is an arbitrary function. To determine it, differentiate $\phi(x, y)$ with respect to y.

$$\frac{\partial \phi}{\partial y} = 2(y^2 + y + x - 1)e^{x+2y} + f'(y)$$

In order for this equation to be consistent with equation (3), we require that $f'(y) = 0$, which means $f(y) = B$, a constant. Consequently,

$$\phi(x, y) = (x - 1 + y^2)e^{x+2y} + B.$$

So for the general solution to the ODE, we have

$$(x - 1 + y^2)e^{x+2y} + B = A.$$

Subtract B from both sides and introduce a new arbitrary constant C. Therefore,

$$(x - 1 + y^2)e^{x+2y} = C.$$